

ON BRAUER p -DIMENSIONS AND ABSOLUTE BRAUER p -DIMENSIONS OF HENSELIAN FIELDS

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ABSTRACT. This paper studies the Brauer p -dimension $\text{Brd}_p(K)$ and the absolute Brauer p -dimension $\text{abrd}_p(K)$ of a field K with a Henselian valuation v , for a prime number p different from the characteristic of the residue field \widehat{K} of (K, v) . Our main result allows us to find exact formulae for $\text{Brd}_p(K)$ and $\text{abrd}_p(K)$ in a number of interesting special cases. It is also used for describing the sequences $(\text{Brd}_p(E), \text{abrd}_p(E))$, indexed by the set of prime numbers, when E runs across the class of Henselian fields, such that $\text{char}(\widehat{E}) = 0$ and the absolute Galois group $\mathcal{G}_{\widehat{E}}$ is of 2-cohomological dimension zero. A similar result is obtained for maximally complete fields of characteristic $q > 0$.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

Let E be a field, $s(E)$ the class of finite-dimensional associative central simple E -algebras, and $d(E)$ the subclass of division algebras $D \in d(E)$. For each $A \in s(E)$, denote by $[A]$ the equivalence class of A in the Brauer group $\text{Br}(E)$, and by D_A some representative of $[A]$ lying in $d(E)$. The existence of D_A and its uniqueness, up-to an E -isomorphism, follows from Wedderburn's structure theorem, and it is known that $\text{Br}(E)$ is an abelian torsion group, so it decomposes into the direct sum of its p -components $\text{Br}(E)_p$, where p runs across the set \mathbb{P} of prime numbers (see [38], Sects 3.5 and 14.4). Recall also that, for each $A \in s(E)$, the dimension $[A: E]$ is a square of a positive integer $\deg(A)$ (the degree of A). The main numerical invariants of both D_A and $[A]$ are the exponent $\exp(A)$, i.e. the order of $[A]$ in $\text{Br}(E)$, and the Schur index $\text{ind}(A) = \deg(D_A)$. Their relations and behaviour under scalar extensions of finite degrees are described as follows (cf. [38], Sects. 13.4, 14.4 and 15.2):

(1.1) (i) $\exp(A)$ divides $\text{ind}(A)$ and is divisible by every $p \in \mathbb{P}$ dividing $\text{ind}(A)$. For each $B \in s(E)$ with $\text{ind}(B)$ relatively prime to $\text{ind}(A)$, $\text{ind}(A \otimes_E B) = \text{ind}(A) \cdot \text{ind}(B)$; in particular, the tensor product $A \otimes_E B$ lies in $d(E)$, provided that $A \in d(E)$ and $B \in d(E)$;

(ii) $\text{ind}(A)$ and $\exp(A \otimes_E R)$ divide $\text{ind}(A \otimes_E R)[R: E]$ and $\exp(A)$, respectively, for each finite field extension R/E of degree $[R: E]$.

As shown by Brauer (see, e.g., [38], Sect. 19.6, the comment to [9], Corollary 3.7, and the references there), (1.1) (i) and the following result determine all generally valid relations between Schur indices and exponents:

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(1.2) There exists a field F , such that $d(F)$ contains an algebra $D_{m,n}$ with $\exp(D_{m,n}) = m$ and $\text{ind}(D_{m,n}) = n$, whenever m and n are positive integers with $m \mid n$, which share a common set of prime divisors.

On the other hand, it is known that the relations between Schur indices and exponents over a number of frequently used fields are subject to much tougher restrictions than those described by (1.1) (i) and (1.2). The Brauer p -dimensions $\text{Brd}_p(E)$, $p \in \mathbb{P}$, of a field E and their supremum $\text{Brd}(E)$, the Brauer dimension of E , are invariants which contain an essential information about the additional restrictions on the relations between $\text{ind}(A)$ and $\exp(A)$, $A \in s(E)$, arising as a result of specific properties of E . For convenience of the reader, recall that E is said to be of Brauer p -dimension $\text{Brd}_p(E) = n$, where $n \in \mathbb{Z}$, if n is the least integer ≥ 0 for which $\text{ind}(D) \leq \exp(D)^n$ whenever $D \in d(E)$ and $[D] \in \text{Br}(E)_p$. In view of (1.1), this means that $\text{Brd}(E) \leq 1$ if and only if E is a stable field (in the sense of Risman, [42]), i.e. $\text{ind}(D) = \exp(D)$, for each $D \in d(E)$. Note also that $\text{Brd}_p(E) = 0$, for a given $p \in \mathbb{P}$, if and only if $\text{Br}(E)_p = \{0\}$; in particular, $\text{Brd}(E) = 0$ if and only if $\text{Br}(E) = \{0\}$.

We say that $\text{Brd}_p(E) = \infty$, if there exists a sequence $D_\nu \in d(E)$, $\nu \in \mathbb{N}$, such that $[D_\nu] \in \text{Br}(E)_p$ and $\text{ind}(D_\nu) > \exp(D_\nu)^\nu$, for each index ν . By an absolute Brauer p -dimension of E , we mean the supremum $\text{abrd}_p(E) = \sup\{\text{Brd}_p(R) : R \in \text{Fe}(E)\}$. The absolute Brauer dimension $\text{abrd}(E)$ of E is defined as the supremum $\text{abrd}(E) = \sup\{\text{abrd}_p(E) : p \in \mathbb{P}\}$. It is easy to deduce from the Merkurjev-Suslin theorem [36], (16.1), and [7], Lemma 3.5, that if $p \neq \text{char}(E)$ and E_{sep} is a separable closure of E , then $\text{abrd}_p(E)$ is at most equal to the supremum $c_p(E)$ of the symbol lengths $l_R(p, p)$ (defined, e.g., in [3], Sect. 3), when R runs across the set of fixed fields of the open subgroups of some Sylow pro- p -subgroup G_p of the absolute Galois group \mathcal{G}_E , i.e. of the Galois group $\mathcal{G}(E_{\text{sep}}/E)$. These fixed fields contain a primitive p -th root of unity, so $c_p(E)$ is determined by cohomological properties of open subgroups of G_p (cf. [46], page 265, and [36], (11.5)). In particular, it turns out that $\text{abrd}_p(E) < \infty$ if and only if $c_p(E) < \infty$ (see [3], page 230), and $\text{abrd}_p(E) \leq 1$ if and only if $c_p(E) = 1$. Dropping the assumption that $p \neq \text{char}(E)$, note that if $\text{abrd}_p(E) = 0$, then the cohomological p -dimension $\text{cd}_p(\mathcal{G}_E)$ is ≤ 1 , and the converse holds in case E is perfect (see [21], Theorem 6.1.8, or [44], Ch. II, 3.1). Recall further that $\text{Brd}_p(E) = \text{abrd}_p(E) = 1$, for all $p \in \mathbb{P}$, in the following cases: (i) E is a global or local field (see [41], (31.4) and (32.19)); (ii) E is the function field of an algebraic surface defined over an algebraically closed field E_0 [24], [31]. Similarly, if E is the function field of an algebraic curve defined over a perfect PAC-field E_0 , then $\text{Brd}_p(E) = \text{abrd}_p(E) = \text{cd}_p(E_0)$, for each $p \in \mathbb{P}$ [14].

The field E is said to be virtually perfect, if $\text{char}(E) = 0$ or $\text{char}(E) = q > 0$ and E is a finite extension of its subfield $E^q = \{e^q : e \in E\}$. When this holds, it follows from (1.1) (ii) and [1], Ch. VII, Theorem 28, that $\text{abrd}_q(E) \leq \log_q[E : E^q]$. The defined notion and the knowledge of the system $\text{abrd}_p(E)$, $p \in \mathbb{P}$, turn out to be very useful for studying the invariants $\text{Brd}_p(F)$, $p \in \mathbb{P}$, of an arbitrary finitely-generated extension F of E [7].

Fields of finite absolute Brauer p -dimensions, for all $p \in \mathbb{P}$, make interest because the study of their locally finite-dimensional central division algebras

frequently reduces to the finite-dimensional case (see [7], Proposition 1.1, and the reference to its proof). This fact and the preceding observations raise interest in the open problem of whether the class of fields E of finite absolute Brauer p -dimensions, for a fixed $p \in \mathbb{P}$ different from $\text{char}(E)$, is closed under the formation of finitely-generated extensions.

The purpose of this paper is to shed light on the behaviour of Brauer p -dimensions and absolute Brauer p -dimensions by proving the following:

Theorem 1.1. *Let $(\bar{a}, \bar{b}) = a_p, b_p$: $p \in \mathbb{P}$, be a sequence of elements of $\mathbb{N} \cup \{0, \infty\}$, such that $a_p \geq b_p$, for each p , and let $\Pi(\bar{a}, \bar{b}) = \{p \in \mathbb{P}: a_p = b_p\}$. Suppose that if $2 \notin \Pi(\bar{a}, \bar{b})$, then $a_2 \leq 2b_2$. Then there exists a field E with $(\text{abrd}_p(E), \text{Brd}_p(E)) = (a_p, b_p)$, for each $p \in \mathbb{P}$. Moreover, E can be chosen to be of characteristic $q > 0$, provided that $b_p > 0$, $p \in \mathbb{P} \setminus \Pi(\bar{a}, \bar{b})$, $a_q = b_q$ or $a_q = b_q + 1 < \infty$, and $a_{p'} \leq 2b_{p'}$ in case $b_{p'} \neq \infty$ and $p' \mid (q-1)$.*

Theorem 1.2. *Let $(\bar{a}, \bar{b}) = a_p, b_p$, $p \in \mathbb{P}$, be a sequence of elements of $\mathbb{N} \cup \{0, \infty\}$, such that $a_p \geq b_p$, for each p , and let $\Pi(\bar{a}, \bar{b}) = \{p \in \mathbb{P}: a_p = b_p\}$ and $\Pi_j(\bar{a}, \bar{b})$, $j = 0, 1$, be subsets of \mathbb{P} satisfying the following conditions:*

- (i) $2 \in \Pi_1(\bar{a}, \bar{b})$, $\Pi_1(\bar{a}, \bar{b}) \cap \Pi_0(\bar{a}, \bar{b}) = \emptyset$, and $a_{p_1} = 2b_{p_1} + 1 < \infty$, for each $p_1 \in \Pi_1(\bar{a}, \bar{b})$;
- (ii) $a_{p_0} \leq 2b_{p_0} < \infty$, for each $p_0 \in \Pi_0(\bar{a}, \bar{b}) \setminus \Pi(\bar{a}, \bar{b})$;
- (iii) $\Pi_1(\bar{a}, \bar{b})$ equals the set of those $p'_1 \in \mathbb{P}$, which divide at least one of the numbers $p'_0 - 1$, when p'_0 runs across $\Pi_0(\bar{a}, \bar{b}) \cup \Pi_1(\bar{a}, \bar{b})$.

Then there is a field E with $(\text{abrd}_p(E), \text{Brd}_p(E)) = (a_p, b_p)$, for each $p \in \mathbb{P}$.

In Section 5 we show that the fields whose existence is claimed by Theorems 1.1 and 1.2 can be found in the class of Henselian (valued) fields. Our proof relies on Galois theory and the following result:

Theorem 1.3. *Let (K, v) be a Henselian field with a residue field \widehat{K} satisfying the conditions $\text{char}(\widehat{K}) = q \geq 0$ and $\text{Brd}_p(\widehat{K}) < \infty$, for some $p \in \mathbb{P}$, $p \neq q$. Let also $\tau(p)$ be the dimension of the group $v(K)/pv(K)$ as a vector space over the field \mathbb{F}_p with p elements, ε_p a primitive p -th root of unity in \widehat{K}_{sep} , $\widehat{K}(p)$ the maximal p -extension of \widehat{K} in \widehat{K}_{sep} , $r_p(\widehat{K})$ the rank of $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ as a pro- p -group, and $m_p = \min\{\tau(p), r_p(\widehat{K})\}$. Then:*

- (i) $\text{Brd}_p(K) = \infty$ if and only if $m_p = \infty$ or $\tau(p) = \infty$ and $\varepsilon_p \in \widehat{K}$;
- (ii) $\min\{\text{Brd}_p(\widehat{K}) + \lceil \tau(p)/2 \rceil, \lceil (\tau(p) + m_p)/2 \rceil\} \leq \text{Brd}_p(K) \leq \text{Brd}_p(\widehat{K}) + \lceil (\tau(p) + m_p)/2 \rceil$, provided that $\tau(p) < \infty$ and $\varepsilon_p \in \widehat{K}$;
- (iii) When $m_p < \infty$ and $\varepsilon_p \notin \widehat{K}$, $m_p \leq \text{Brd}_p(K) \leq \text{Brd}_p(\widehat{K}) + m_p$.

The present research was initially motivated by the following special case of Theorem 1.1, proved in [7]:

(1.3) For each $(q, k) \in \mathbb{P} \cup \{0\} \times \mathbb{N}$, there exists a field $E_{q,k}$ with $\text{char}(E) = q$, $\text{Brd}(E_q) = k$ and $\text{abrd}_p(E_{q,k}) = \infty$, for all $p \in \mathbb{P} \setminus P_q$, where $P_0 = \{2\}$ and $P_q = \{p \in \mathbb{P}: p \mid q(q-1)\}$ in case $q \in \mathbb{P}$. If $q > 2$ or $(q, k) = (2, 1)$, then $E_{q,k}$ can be chosen so that $[E_{q,k}: E_{q,k}^q] = \infty$ and $\text{abrd}_q(E_{q,k}) = 0$.

In the setting of (1.3), it follows from [7], Theorem 1.1, that if $E_{q,k}$ is not virtually perfect and $P'_q = P_q \setminus \{q\}$, then $\text{Brd}_p(F) = \infty$, $p \in \mathbb{P} \setminus P'_q$, for every finitely-generated transcendental extension $F/E_{q,k}$. This solves negatively [3], Problem 4.4, by proving that the class of fields of finite Brauer dimensions is not closed under the formation of finitely-generated extensions.

The basic notation and terminology used and conventions kept in this paper are standard, like those in [5] and [6]. The notions of an inertial, a nicely semi-ramified (abbr, NSR), and a totally ramified division K -algebra, where (K, v) is a Henselian field, are defined as in [23]. By a Pythagorean field, we mean a formally real field whose set of squares is additively closed. As usual, $[r]$ stands for the integral part of any real number $r \geq 0$. We use the abbreviation NMM-group, for a nonnilpotent Miller-Moreno group, a nonnilpotent finite group whose proper subgroups are abelian. For each profinite group G , $\text{cd}(G)$ denotes the cohomological dimension of G , $\Phi(G)$ is the topological Frattini subgroup of G , i.e. the intersection of its maximal open subgroups, and $P(G)$ is the set of those $p \in \mathbb{P}$, for which the cohomological p -dimension $\text{cd}_p(G)$ of G is nonzero. We say that G is a Frattini cover of a profinite group H , if H is a homomorphic image of G whose kernel is included in $\Phi(G)$. Throughout, Galois groups are viewed as profinite with respect to the Krull topology, the considered profinite group products are topological, and by a profinite group homomorphism, we mean a continuous one. The reader is referred to [21], [29], [15], [23], [38] and [44], for any missing definitions concerning field extensions, orderings and valuation theory, simple algebras, Brauer groups and Galois cohomology.

The paper is organized as follows: Section 2 includes preliminaries on Henselian fields used in the sequel, and Galois-theoretic ingredients of the proof of Theorem 1.1. Theorem 1.3 is proved in Section 3, and is used in Sections 4 and 5 for finding the Brauer and the absolute Brauer p -dimensions of Henselian fields with residue fields of several interesting types, such as global, local, algebraically closed or real closed. Theorems 1.1 and 1.2 are proved in Section 6. We also show there that the set of sequences (a_p, b_p) , $p \in \mathbb{P}$, singled out by Theorem 1.1 or 1.2 coincides with the set of sequences $(\text{abrd}_p(K), \text{Brd}_p(K))$, $p \in \mathbb{P}$, when K runs across the class of Henselian fields with $\text{char}(\widehat{K}) = 0$ and $\text{cd}(\mathcal{G}_{\widehat{K}}) \leq 1$. As a major step in this direction, we find lower and upper bounds for $\text{abrd}_p(K)$, for a Henselian field (K, v) with $\text{abrd}_p(\widehat{K}) < \infty$; we assume that (K, v) is maximally complete in case $p = \text{char}(K)$, i.e. that it does not admit immediate valued proper extensions.

2. Preliminaries on Henselian valuations and Galois theory

A Krull valued field (K, v) is said to be Henselian, if v is uniquely, up-to an equivalence, extendable to a valuation v_L on each algebraic extension L/K . When this holds, we denote by $v(L)$ the value group and by \widehat{L} the residue field of (L, v_L) . It is well-known that if (K, v) is Henselian, then \widehat{L}/\widehat{K} is an algebraic extension and $v(K)$ is a subgroup of $v(L)$. Moreover, Ostrowski's theorem states the following (cf. [15], Theorem 17.2.1):

(2.1) If L/K is finite and $e(L/K)$ is the index of $v(K)$ in $v(L)$, then $[\widehat{L} : \widehat{K}]e(L/K)$ divides $[L : K]$ and $[L : K][\widehat{L} : \widehat{K}]^{-1}e(L/K)^{-1}$ is not divisible by any $p \in \mathbb{P}$, $p \neq \text{char}(\widehat{K})$; when $\text{char}(\widehat{K}) \nmid [L : K]$, $[L : K] = [\widehat{L} : \widehat{K}]e(L/K)$.

It is crucial for our further considerations that when (K, v) is Henselian, each $\Delta \in d(K)$ has a unique, up-to an equivalence, valuation v_Δ extending v so that the value group $v(\Delta)$ of (Δ, v_Δ) is abelian (cf. [48] and [19]). It is known that $v(K)$ is a subgroup of $v(\Delta)$ of index $e(\Delta/F) \leq [\Delta : K]$, the residue division ring $\widehat{\Delta}$ of (Δ, v_Δ) is a \widehat{K} -algebra, and $[\widehat{\Delta} : \widehat{K}] \leq [\Delta : K]$. Moreover, the Ostrowski-Draxl theorem [12], extends (2.1) as follows:

(2.2) $e(\Delta/K)[\widehat{\Delta} : \widehat{K}]$ divides $[\Delta : K]$; if $\text{char}(\widehat{K}) \nmid \text{ind}(\Delta)$, then $[\Delta : K] = e(\Delta/K)[\widehat{\Delta} : \widehat{K}]$.

Statement (2.1) and the Henselity of (K, v) imply the following:

(2.3) The quotient groups $v(K)/pv(K)$ and $v(L)/pv(L)$ are isomorphic, provided that $p \in \mathbb{P}$ and L/K is a finite extension. Moreover, if $\text{char}(\widehat{K}) \nmid [L : K]$, then the natural embedding of K into L induces canonically an isomorphism $v(K)/pv(K) \cong v(L)/pv(L)$.

A finite extension R of K is said to be defectless, if $[R : K] = [\widehat{R} : \widehat{K}]e(R/K)$, and it is called inertial, if $[R : K] = [\widehat{R} : \widehat{K}]$ and \widehat{R} is separable over \widehat{K} . We say that R/K is totally ramified, if $[R : K] = e(R/K)$. The extension R/K is called tamely ramified, if \widehat{R}/\widehat{K} is separable and $\text{char}(\widehat{K}) \nmid e(R/K)$. Let K_{ur} be the compositum of inertial extensions of K in K_{sep} . It is known that K_{ur}/K is a Galois extension, \widehat{K}_{ur} is a separable closure of \widehat{K} and $v(K_{\text{ur}}) = v(K)$. Similarly, the compositum K_{tr} of tamely ramified extensions of K in K_{sep} is a Galois extension of K , such that $v(K_{\text{tr}}) = pv(K_{\text{tr}})$, for all $p \in \mathbb{P}$, $p \neq \text{char}(\widehat{K})$. It is therefore clear from (2.1) that if $K_{\text{tr}} \neq K_{\text{sep}}$, then $\text{char}(\widehat{K}) = q \neq 0$ and $\mathcal{G}_{K_{\text{tr}}}$ is a pro- q -group. When this holds, the Mel'nikov-Tavgen' theorem [34], combined with (2.1), (2.3) and Galois theory, implies the existence of a field $K' \in I(K_{\text{sep}}/K)$ satisfying the following:

(2.4) $K' \cap K_{\text{tr}} = K$, $K'K_{\text{tr}} = K_{\text{sep}}$ and K_{sep} is K -isomorphic to $K_{\text{tr}} \otimes_K K'$; the field \widehat{K}' is a perfect closure of \widehat{K} , finite extensions of K in K' are of q -primary degrees, $K_{\text{sep}} = K'_{\text{tr}}$, $v(K') = qv(K')$, and the natural embedding of K into K' induces isomorphisms $v(K)/pv(K) \cong v(K')/pv(K')$, $p \in \mathbb{P} \setminus \{q\}$.

The following lemma has been proved in [7], Sect. 2.

Lemma 2.1. *Let K_0 be a perfect field of characteristic $q \geq 0$, and let $n(p)$: $p \in \mathbb{P}$, be a sequence of elements of $\mathbb{N} \cup \{0, \infty\}$. Then there exists a Henselian field (K, v) , such that $\text{char}(K) = q$, $\widehat{K} = K_0$, and for each $p \in \mathbb{P}$, the group $v(K)/pv(K)$ has dimension $n(p)$ as a vector space over \mathbb{F}_p . Moreover, if $q > 0$, then K can be chosen so that $[K : K^q] = n(q)$ and finite extensions of K be defectless relative to v .*

Lemma 2.1 and our next lemma play a crucial role in the proofs of (1.3) and Theorems 1.1 and 1.2.

Lemma 2.2. *Assume that $c_p: p \in \mathbb{P}$, is a sequence of positive integers, such that c_p divides $p-1$, for each p , and P is a subset of \mathbb{P} including the set Π of those $\pi \in \mathbb{P}$, for which there exists $p_\pi \in \mathbb{P}$ with c_{p_π} divisible by π . Then there exists a field E_0 with $\text{char}(E_0) = 0$, \mathcal{G}_{E_0} isomorphic to the group product $\mathbb{Z}_P = \prod_{p \in P} \mathbb{Z}_p$, and $[E_0(\varepsilon_p): E_0] = c_p$, where ε_p is a primitive p -th root of unity in $E_{0,\text{sep}}$.*

Proof. Let \mathbb{Q} be the field of rational numbers, and ε_p a primitive p -th root of unity in \mathbb{Q}_{sep} , for each $p \in \mathbb{P}$. It is well-known (cf. [29], Ch. VIII, Sect. 3) that $[\mathbb{Q}(\varepsilon_p): \mathbb{Q}] = p-1$ and the extension $\mathbb{Q}(\varepsilon_p)/\mathbb{Q}$ is cyclic, so it follows from Galois theory that there exists $\Phi_p \in I(\mathbb{Q}(\varepsilon_p)/\mathbb{Q})$ with $[\Phi_p: \mathbb{Q}] = (p-1)/c_p$, for each $p \in \mathbb{P}$. Denote by Φ and Φ' the compositums of the fields Φ_p , $p \in \mathbb{P}$, and $\Phi_p(\varepsilon_p)$, $p \in \mathbb{P}$, respectively, and put $\Theta_p = \Phi(p) \cap \Phi'$, for each p . It is clear from Galois theory, the definition of Φ , and the irreducibility of cyclotomic polynomials over \mathbb{Q} that $\Phi(\varepsilon_p)\Psi_p = \Phi'$ and $\Phi(\varepsilon_p) \cap \Psi_p = \Phi$, for each $p \in \mathbb{P}$, where Ψ_p is the compositum of the fields $\Phi(\varepsilon_{\bar{p}})$, $\bar{p} \in \mathbb{P} \setminus \{p\}$. This implies $\Phi(\varepsilon_p)/\Phi$ and Φ'/Ψ_p are degree c_p cyclic extensions, $p \in \mathbb{P}$, and Φ'/Φ is a Galois extension with $\mathcal{G}(\Phi'/\Phi)$ isomorphic to $\prod_{p \in \mathbb{P}} \mathcal{G}(\Phi(\varepsilon_p)/\Phi)$. Similarly, it is proved that Θ_p/Φ is Galois with $\mathcal{G}(\Theta_p/\Phi) \cong \prod_{p' \in \mathbb{P}} C_{p,p'}$, for every $p \in \mathbb{P}$, where $C_{p,p'}$ is the Sylow p -subgroup of $\mathcal{G}(\Phi(\varepsilon_p)/\Phi)$ when p' runs across \mathbb{P} . This ensures that the continuous character group $C(\Theta_p/\Phi)$ of $\mathcal{G}(\Theta_p/\Phi)$ is isomorphic to the direct sum $\oplus_{p' \in \mathbb{P}} C_{p,p'}$ (cf. [25], Ch. 7, Sect. 5). The obtained result indicates that there exists a homomorphism y_p of $C(\Theta_p/\Phi)$ into the quasicyclic p -group $\mathbb{Z}(p^\infty)$ satisfying the following:

- (2.5) (i) y_p is surjective, if the exponent θ_p of $C(\Theta_p/\Phi)$ is infinite; the image of y_p equals the subgroup of order θ_p in $\mathbb{Z}(p^\infty)$, in case θ_p is finite;
- (ii) y_p maps injectively the direct summands $C_{p,p'}$, $p' \in \mathbb{P}$, into $\mathbb{Z}(p^\infty)$.

In view of Galois theory and Pontrjagin's duality (cf. [25], Ch. 7, Sect. 5), (2.5) can be restated as follows:

- (2.6) There exists a field $Y_p \in I(\Theta_p/\Phi)$, such that $\mathcal{G}(\Theta_p/Y_p)$ is procyclic and $Y_p \cap \Phi(\varepsilon_{p'}) = \Phi$, for each $p' \in \mathbb{P}$.

Let now Y be the compositum of the fields Y_p , $p \in \mathbb{P}$. Then $Y \in I(\Phi'/\Phi)$ and it follows from (2.6), Galois theory and the structure of the groups $\mathcal{G}(\Phi'/\Phi)$ and $\mathcal{G}(\Theta_p/\Phi)$, $p \in \mathbb{P}$, that Φ'/Y is a Galois extension, such that $\mathcal{G}(\Phi'/Y)$ is procyclic and $[Y(\varepsilon_p): Y] = c_p$, for each $p \in \mathbb{P}$. Put $Y' = \Phi'$, provided that $\Pi = P$, and denote by Y' the compositum of Φ' of the fields Γ_p , $p \in \mathbb{P} \setminus \Pi$, where Γ_p is the cyclotomic \mathbb{Z}_p extension of \mathbb{Q} in \mathbb{Q}_{sep} , for each $p \in \mathbb{P}$ (defined, e.g., in [30], Ch. 5). Consider the set $\Omega(\Phi)$ of those nonreal fields $R \in I(\mathbb{Q}_{\text{sep}}/\Phi)$, for which $R \cap E' = \Phi$. It follows from Galois theory and well-known properties of cyclotomic extensions of \mathbb{Q} that $\Phi(\sqrt{-1}) \in \Omega(\Phi)$; in particular, $\Omega(\Phi)$ is nonempty. Note also that $\Omega(\Phi)$ satisfies the conditions of Zorn's lemma with respect to the partial ordering by inclusion, so it contains a maximal element E_0 . Applying Galois theory, one obtains from the condition on E_0 that \mathcal{G}_{E_0} is a procyclic group with $P(\mathcal{G}_{E_0}) = P$. Since E_0 is nonreal, it is now easy to deduce from [52], Theorem 2, that the Sylow pro- p -subgroup of \mathcal{G}_{E_0} is isomorphic to \mathbb{Z}_p , for every $p \in P$, as well as to see that E_0 has the properties required by Lemma 2.2. \square

Lemma 2.2 and the following two lemmas present the Galois-theoretic ingredients of the proof of Theorems 1.1 and 1.2.

Lemma 2.3. *Let E_0 be a field and L_0 a Galois extension of E_0 . Then there exists a field extension E/E_0 , such that $\text{cd}(\mathcal{G}_E) \leq 1$ and $L = L_0 \otimes_{E_0} E$ is a field with $L \cap R \neq E$, for every $R \in I(L_{\text{sep}}/E)$ different from E .*

Proof. It follows from [20], Proposition 1.2 and Theorem 2.6 (see also [7], Theorem 1.3 (i)) that there exists a field extension E_1/E_0 , such that E_0 is separably closed in E_1 and $\text{Br}(E'_1) = \{0\}$, for each finite separable extension E'_1/E_1 . In particular, this ensures that $\text{cd}(\mathcal{G}_{E_1}) \leq 1$ and the E_1 -algebra $L_1 = L_0 \otimes_{E_0} E_1$ is a field (cf. [44], Ch. II, 3.1, and [21], Theorem 6.1.8). More precisely, it becomes clear that L_1/E_1 is a Galois extension with $\mathcal{G}(L_1/E_1) \cong \mathcal{G}(L_0/E_0)$. Identifying L_1 with its E_1 -isomorphic copy in $E_{1,\text{sep}}$, put $\Sigma = \{Y \in I(E_{1,\text{sep}}/E_1) : Y \cap L_1 = E_1\}$. It is easily verified that Σ is nonempty and possesses a maximal element E with respect to the partial ordering by inclusion. In view of Galois theory, this implies that $L_1 E$ is a Galois extension of E with $\mathcal{G}(L_1 E/E) \cong \mathcal{G}(L_1/E_1) \cong \mathcal{G}(L_0/E_0)$. It is also clear that $R_1 \cap L_1 E \neq E$ whenever $R_1 \in I(E_{1,\text{sep}}/E)$ and $R_1 \neq E$. Observing finally that $L_1 E$ is E -isomorphic to L (cf. [38], Sect. 9.4, Corollary a) and $\text{cd}(\mathcal{G}_E) \leq \text{cd}(\mathcal{G}_{E_1}) \leq 1$, one completes the proof of Lemma 2.3. \square

It is clear from Galois theory that the concluding assertion of Lemma 2.3 can be restated by saying that \mathcal{G}_E is a Frattini cover of $\mathcal{G}(L/E)$. This fact is frequently used in Section 5.

Lemma 2.4. *In the setting of Lemma 2.3, $P(\mathcal{G}(L_0/E_0)) = P(\mathcal{G}_E)$. In addition, \mathcal{G}_E is pronilpotent if and only if so is $\mathcal{G}(L_0/E_0)$.*

Proof. Let Ψ and Ψ_0 be the fixed fields of $\mathcal{G}(L/E)$ and $\mathcal{G}(L_0/E_0)$, respectively. Identifying $E_{0,\text{sep}}$ with its E_0 -isomorphic copy in E_{sep} , one obtains from Galois theory and Lemma 2.3 that $\Psi = \Psi_0 E$ and Ψ is the fixed field of $\Phi(\mathcal{G}_E)$. This implies $\Phi(\mathcal{G}_\Psi)$ is pronilpotent and its Sylow pro- p -subgroup is normal in \mathcal{G}_E , for each $p \in \mathbb{P}$. Similarly, it becomes clear that $\mathcal{G}(L/\Psi)$ is pronilpotent and its Sylow pro- p -subgroups are normal in $\mathcal{G}(L/E)$. In view of the analogue to the Schur-Zassenhaus theorem (cf. [26], Ch. 7, Theorem 20.2.6) for profinite groups, these observations enable one to prove the inclusion $P(\Phi(\mathcal{G}_E)) \cup P(\mathcal{G}(L_0/\Psi_0)) \subseteq P(\mathcal{G}(\Psi/E))$ by assuming the opposite. It is now easy to see that $P(\mathcal{G}(\Psi_0/E_0)) = P(\mathcal{G}(\Psi/E)) = P(\mathcal{G}(L_0/E_0)) = P(\mathcal{G}_E)$, which proves the former assertion of the lemma. Similarly, it follows from the variant of the Burnside-Wielandt theorem for profinite groups (cf. [26], Ch. 6, Theorem 17.1.4) that $\mathcal{G}(L_0/E_0)$ and \mathcal{G}_E is pronilpotent if and only if so is $\mathcal{G}(\Psi/E)$, so Lemma 2.4 is proved. \square

Arguing as in the proof of the main result of [51], one obtains the following:

(2.7) Given a field E_0 and a profinite group H , there exists a purely transcendental extension E'/E and a field $E \in I(E'/E)$, such that E'/E is a Galois extension with $\mathcal{G}(E'/E) \cong H$. Hence, for each Galois extension

L_0/E_0 , $L_0 \otimes_{E_0} E' = L'_0$ is a field that is a Galois extension of E with $\mathcal{G}(L'_0/E) \cong \mathcal{G}(L_0/E_0) \times H$.

The following well-known fact (cf., e.g., [40], Theorem 445) will be used in Section 5 for proving Theorem 1.1 in the case where $0 < a_p = 2b_p < \infty$ and $p = 2$ or $p \mid (q - 1)$:

(2.8) For each pair $(\pi, p) \in \mathbb{P}^2$ with $\pi \neq p$, if k is the order of the coset $\pi + p\mathbb{Z}$ in the multiplicative group of $\mathbb{Z}/p\mathbb{Z}$, then there exists a nonabelian group $G_{\pi,p;k}$ of order $\pi^k p$ whose Sylow π -subgroup $H_{\pi,p;k}$ is a minimal normal subgroup of $G_{\pi,p;k}$; in particular, $H_{\pi,p;k}$ is abelian of exponent π .

When E_0 is a field containing a primitive π -th root of unity, L_0/E_0 is a degree p cyclic extension, φ is a generator of $\mathcal{G}(L_0/E_0)$, λ is a primitive element of L_0/E_0 , and $L' = L_0(X)$ is a rational function field in one indeterminate over L_0 , one deduces the following statement from Kummer theory and Maschke's theorem:

(2.9) The extension of L' in L'_{sep} generated by the π -th roots of $(X - \varphi^j(\lambda))$, $j = 0, 1, \dots, p - 1$, contains as a subfield a Galois extension L of $E_0(X)$, such that $L' \subset L$ and $\mathcal{G}(L/E_0(X)) \cong G_{\pi,p;k}$.

The existence of a Galois extension $L/E_0(X)$ with the properties required by (2.9) follows from the Artin-Schreier theorem (cf. [29], Ch. VIII, Sect. 6) and Maschke's theorem, if E_0 is a field of characteristic π , L_0/E_0 is a cyclic extension of degree p , φ , λ and L' are defined as above, and we consider the extension of L' generated by the roots in L'_{sep} of the polynomials $Y^p - Y - X - \varphi^j(\lambda) \in L'[Y]$, $j = 0, 1, \dots, p - 1$.

3. Proof of Theorem 1.3

The purpose of this Section is to determine $\text{Brd}_p(K)$, for a Henselian field (K, v) with $\text{char}(\widehat{K}) \neq p$. Our starting point are the following results of [23]:

(3.1) (i) If $D \in d(K)$ and $\text{char}(\widehat{K}) \nmid \text{ind}(D)$, then D is Brauer equivalent to $S \otimes_K V \otimes_K T$, for some S, V and $T \in d(K)$, such that D/K is inertial, V/K is NSR, and for each $n \in \mathbb{N}$, the algebra $T_n \in d(K)$ representing the class $n[T]$ is totally ramified over K ;

(ii) The set $\text{IBr}(K)$ of Brauer equivalence classes of inertial K -algebras $S' \in d(K)$ forms a subgroup of $\text{Br}(K)$; the canonical mapping of $\text{IBr}(K)$ into $\text{Br}(\widehat{K})$ is a group isomorphism; in particular, if $\text{Brd}_p(\widehat{K}) = \infty$, for some $p \in \mathbb{P}$, then $\text{Brd}_p(K) = \infty$;

(iii) If $T' \in d(K)$ is a totally ramified K -algebra and $\text{char}(\widehat{K}) \nmid \text{ind}(T')$, then K contains a primitive root of unity of degree equal to $\exp(T')$; in addition, if $T' \neq K$, then $v(K)/pv(K)$ is a noncyclic group.

In what follows, we will also need the following result:

(3.2) If $n \in \mathbb{N}$ and D, S, V and T are related as in (3.1) (i), then $\text{IBr}(K)$ contains the class $n[D]$ if and only if n is divisible by $\exp(V)$ and $\exp(T)$.

It follows from the structure of V [23], Theorem 4.4, and the theory of cyclic algebras (see [38], Sect. 15.1, Corollary b) that the algebra $V_n \in d(K)$

representing the class $n[V]$ is NSR, for each $n \in \mathbb{N}$. Therefore, (3.2) can be deduced from the concluding part of (3.1) (i) and the following result:

(3.3) Let D , S , V and T be related as in (3.1) (i). Then:

- (i) D/K is inertial if and only if $V = T = K$; D/K is inertially split, i.e. $[D] \in \text{Br}(K_{\text{ur}}/K)$, if and only if $T = K$;
- (ii) $\exp(D) = \text{lcm}(\exp(\widehat{S}), \exp(V), \exp(T))$.

The right-to-left implications in (3.3) (i) are obvious and the necessity in the latter part of (3.3) (i) follows from [23], Corollary 3.5. Therefore, it suffices to prove the necessity in the former part of (3.3) (i), under the extra hypothesis that $T = K$. Then our assertion follows from [23], Exercise 4.3 and Theorem 4.4. The obtained result and (3.1) (ii) imply that if $\nu \in \mathbb{Z}$, then $\nu[D] = 0$ if and only if $\nu[S] = \nu[V] = \nu[T] = 0$, which proves (3.3) (ii).

The rest of this Section is devoted to the proof of Theorem 1.3. Our argument relies on the structure of NSR-algebras determined by the following lemma (for a proof, see [23], Theorem 4.4).

Lemma 3.1. *Let (K, v) be a Henselian field and V an NSR-algebra over K . Then V is isomorphic to the K -algebra $V_1 \otimes_K \cdots \otimes_K V_\nu$, where $\nu \in \mathbb{N}$, and for each index i , $V_i \in d(K)$, $[V_i] \in \text{Br}(K)_p$ and V_i is a cyclic NSR-algebra over K . In other words, for each i , there is an inertial cyclic extension U_i/K and an element $\pi_i \in K^*$, such that $[U_i: K] = \text{ind}(V_i)$ and $(U_i/K, \sigma_i, \pi_i) \cong V_i$, where σ_i is a generator of $\mathcal{G}(U_i/K)$, and the subgroup $W(V)$ of $v(K)/pv(K)$, generated by the cosets $v(\pi_i) + pv(K)$, $i = 1, \dots, \nu$, is of order p^ν .*

Lemma 3.1 and [38], Sect. 15.1, Proposition b, imply that $\exp(V_i) = \text{ind}(V_i)$, $i = 1, \dots, \nu$, $\exp(V) = \max\{\text{ind}(V_i): i = 1, \dots, \nu\}$, and $\text{ind}(V) \mid \exp(V)^\nu$. It also follows from Lemma 3.1 that the K -subalgebra $U = U_1 \otimes_K \cdots \otimes_K U_\nu$ of V is a field. More precisely, U/K is an inertial Galois extension with $\mathcal{G}(U/K)$ isomorphic to the direct sum of ν nontrivial cyclic p -groups. Hence, by the Henselian property of (K, v) , \widehat{U}/\widehat{K} is also Galois with $\mathcal{G}(\widehat{U}/\widehat{K}) \cong \mathcal{G}(U/K)$, which yields $\nu \leq r_p(\widehat{K})$. Since $W(V)$ is of order p^ν , this proves that $\nu \leq m_p$. It is now easy to deduce from (3.1) (i) and Lemma 3.1 that if D is inertially split, then $\text{ind}(D) \mid p^m \cdot w(p)$, where $w(p) = \text{Brd}_p(\widehat{K}) + r_p(\widehat{K})$.

Conversely, it is known (cf. [23], Example 4.3) that, for each $\nu' \in \mathbb{N}$, $\nu' \leq m_p$, there exists an NSR-algebra $V' \in d(K)$, such that $\exp(V') = p$ and $\text{ind}(V') = p^{\nu'}$. In view of (3.1) (iii), the obtained result completes the proof of Theorem 1.3 in the case where $\varepsilon_p \notin \widehat{K}$ or $m_p = \infty$. For the rest of our proof, we need the following lemma.

Lemma 3.2. *Let (K, v) be a Krull valued field containing a primitive p -th root of unity ε , for a given $p \in \mathbb{P}$ not equal to $\text{char}(\widehat{K})$, and suppose that there exist elements $\alpha_1, \dots, \alpha_{2n} \in K^*$, for some $n \in \mathbb{N}$, such that the cosets $v(\alpha_i) + pv(K)$, $i = 1, \dots, 2n$, generate a subgroup of $v(K)/pv(K)$ of order p^{2n} . Denote by Δ_i the symbol K -algebra $A_\varepsilon(\alpha_{2i-1}, \alpha_{2i}; K)$, for each index $i \leq n$, and put $D_n = \otimes_{i=1}^n \Delta_i$, where $\otimes = \otimes_K$. Then $D_n \in d(K)$, $\exp(D_n) = p$ and $\text{ind}(D_n) = p^n$.*

Proof. It is clearly sufficient to consider the special case where v is Henselian. Our assumptions show that the polynomials $f_j(X) = X^p - \alpha_j \in K[X]$, $j = 1, \dots, 2n$, are irreducible over K . In view of Kummer theory, this means that the root field $K_j \in \text{Fe}(K)$ of $f_j(X)$ over K is a degree p cyclic extension of K , for each index n . Denote by L_n the compositum $K_1 \dots K_{2n}$. Identifying $v(K)$, $v(L_n)$ and $v(K_j)$, $j = 1, \dots, 2n$, with their natural isomorphic copies in a divisible hull $\overline{V}(K)$ of $v(K)$, one obtains that $v(K_j)$ is generated by $v(K)$ and $(1/p)v(\alpha_j)$, and the sum of the groups $v(K_1)/v(K), \dots, v(K_{2n})/v(K)$ is direct and equal to $v(L_n)/v(K)$. At the same time, for each index j , it follows from the uniqueness of v_{K_j} that $v(\lambda_j) \in pv(K_j)$ whenever $\lambda_j \in N(K_j/K)$. This implies that $\alpha_{2i} \notin N(K_{2i-1}/K)$, so it follows from [38], Sect. 15.1, Proposition b, that $\Delta_i \in d(K)$, $i = 1, \dots, n$, which proves Lemma 3.2 in case $n = 1$. In addition, it follows from (2.2) that $v(\Delta_i) = v(K_{2i-1}) + v(K_{2i})$ and $\widehat{\Delta}_i = \widehat{K}$, for each index $i \leq n$. Observing also that the sum of the groups $v(\Delta_i)/v(K)$, $i = 1, \dots, n$, is direct and equal to $v(L_n)/v(K)$, and applying [37], Theorem 1 (see also [23], page 132), one concludes that $D_n \in d(K)$, $\widehat{D}_n = \widehat{K}$ and $v(D_n) = v(L_n)$, so Lemma 3.2 is proved. \square

Remark 3.3. The conclusion of Lemma 3.2 remains valid, if the former condition of the lemma is replaced by the one that $\text{char}(K) = p > 0$ and Δ_i is the p -symbol K -algebra $[\alpha_{2i-1}, \alpha_{2i}]$ (defined, e.g., in [47]), for $i = 1, \dots, n$. The proof is essentially the same, by working in an algebraic closure \overline{K} of K with the polynomials $f_{2i-1}(X) = X^p - X - \alpha_{2i-1}$ and $f_{2i}(X) = X^p - \alpha_{2i}$, $i = 1, \dots, n$. This yields $\text{Brd}_p(K) = \infty$ when $\tau(p) = \infty$.

Our objective now is to prove Theorem 1.3 in the case where $\varepsilon_p \in \widehat{K}$. As v is Henselian and $p \neq \text{char}(\widehat{K})$, then K contains a primitive p -th root of unity ε , so it follows from Lemma 3.2 that if $\tau(p) = \infty$, then $\text{Brd}_p(K) = \infty$. In the rest of our proof, we assume that $\tau(p) < \infty$, and put $\beta'_p = \lceil (\tau(p) + m_p)/2 \rceil$ and $\beta_p = \text{Brd}_p(\widehat{K}) + \beta'_p$. The inequality $\text{Brd}_p(\widehat{K}) + \lceil \tau(p)/2 \rceil \leq \text{Brd}_p(K)$ is implied by Lemma 3.2 and [37], Theorem 1. In order to complete the proof of Theorem 1.3 it remains to be seen that $\beta'_p \leq \text{Brd}_p(K) \leq \beta_p$. It follows from (3.1) (ii), Lemma 3.2 and [37], Theorem 1, that $\text{Brd}_p(K) = 0$ if and only if $\text{Brd}_p(\widehat{K}) = 0$ and $\tau(p) \leq 1$. When this holds, one obviously has $\beta'_p = \beta_p \leq 1$, so we suppose further that $\text{Brd}_p(K) > 0$. We prove that $\text{Brd}_p(K) \leq \beta_p$ by showing that $\text{ind}(D) \mid p^{m\beta_p}$, for an arbitrary $D \in d(K)$ of exponent p^m , where $m \in \mathbb{N}$. Since, by (1.1) (ii), $\exp(D \otimes_K Y) \mid \exp(D)$ and $\text{ind}(D) \mid \text{ind}(D \otimes_K Y)[Y : K]$, for every finite field extension Y/K , it suffices for the purpose to establish the following fact:

(3.4) There exists a totally ramified field extension Θ/K , such that $[\Theta : K] \mid p^{m\beta'_p}$ and $[D \otimes_K \Theta] \in \text{IBr}(\Theta)$.

Take algebras S , V and $T \in d(K)$ related to D as in (3.1) (i). Note that if D is inertially split, then one can take as Θ any maximal subfield of V , which is totally ramified over K . Suppose further that D is not inertially split, i.e. $T \neq K$, and $\exp(T) = p^t$. Then, by the proof of [23], Lemma 6.2, the structure of T is determined as follows:

(3.5) There exist positive integers μ and t_1, \dots, t_μ , such that $\max\{t_j : j = 1, \dots, \mu\} = t$, $T \cong T_1 \otimes_K \dots \otimes_K T_\mu$, and for each index j , $T_j \in d(K)$, $\text{ind}(T_j) = p^{t_j}$, T_j/K is totally ramified and T_j is K -isomorphic to the symbol algebra $A_{\eta_j}(a_j; b_j; K)$ (of degree p^{t_j}), where η_j is a primitive root of unity in K of degree p^{t_j} . In addition, the cosets $v(a_j) + pv(K)$ and $v(b_j) + pv(K)$, $j = 1, \dots, \mu$, generate a subgroup of order $p^{2\mu}$ in $v(K)/pv(K)$.

In view of Kummer theory and [38], Sect. 15.1, Proposition b, statements (3.5) prove that $\exp(T_j) = \text{ind}(T_j) = p^{t_j}$, $j = 1, \dots, \mu$, and $\text{ind}(T) \mid \exp(T)^\mu$.

We prove (3.4) proceeding by induction on m . Suppose first that $m = 1$ and denote by $W(V)$ the subgroup of K^*/K^{*p} generated by the cosets $\pi_i K^{*p}$, $i = 1, \dots, \nu$, where π_1, \dots, π_ν are defined in Lemma 3.1. Fix a subset $\{\pi'_1, \dots, \pi'_\nu\}$ of K^* so that $\pi'_i K^{*p}$, $i = 1, \dots, \nu$, is an \mathbb{F}_p -basis of $W(V)$. Using [23], Remark 4.6 (a), and the fact that V_1, \dots, V_ν are symbol K -algebras, one proves the existence of fields $U'_i \in I(U/K)$ and p -th roots of unity ξ_i , $i = 1, \dots, \nu$, satisfying the following:

(3.6) $U'_1 \dots U'_\nu = U$, $[U'_i : K] = p$, $i = 1, \dots, \nu$, and there are generators $\sigma'_1, \dots, \sigma'_\nu$ of $\mathcal{G}(U'_1/K), \dots, \mathcal{G}(U'_\nu/K)$, respectively, such that the K -algebra $V'_1 \otimes_K \dots \otimes_K V'_\nu$ is isomorphic to V , where $V'_i = (U'_i/K, \sigma'_i, \xi_i \pi'_i)$, for each i .

The algebra $T \in d(K)$ is considered similarly. It is known (cf. [23]) that the cosets of the elements $v(a_j), v(b_j)$: $j = 1, \dots, \mu$, generate a subgroup $W'(T) \leq v(K)/pv(K)$ of order $p^{2\mu}$. Let now $W^*(T) = \langle a_j, b_j : j = 1, \dots, \mu \rangle \cdot K^{*p}$ and π' be an element of $W^*(T) \setminus K^{*p}$. Using [38], Sect. 15.1, Proposition b, Kummer theory and elementary properties of symbol K -algebras, and arguing similarly to the proof of (3.6), one concludes that:

(3.7) $W^*(T)$ contains elements a'_j, b'_j , $j = 1, \dots, \mu$, such that $v(b'_1) = v(\pi')$ and T is K -isomorphic to $A_\varepsilon(a'_1, b'_1; K) \otimes_K \dots \otimes_K A_\varepsilon(a'_\mu, b'_\mu; K)$, where $a'_j, b'_j \in K$, $a_j''^2 = a_j'^2$ and $b_j''^2 = b_j'^2$, for each index j . When $p > 2$ or $\sqrt{-1} \in K$, the isomorphism holds, for $a_j'' = a_j'$ and $b_j'' = b_j'$, $j = 1, \dots, \mu$.

Now choose V and T so that the sum $\mu + \nu$ be minimal. We show that $V \otimes_K T \in d(K)$ over K , and the cosets modulo $pv(K)$ of the elements of the system $\Psi = \{v(\pi_i), v(a_j), v(b_j) : i = 1, \dots, \mu; j = 1, \dots, \nu\}$ are linearly independent over \mathbb{F}_p . Assuming the opposite, one obtains that the elements π'_i , $i = 1, \dots, \nu$, and a'_j, b'_j , $j = 1, \dots, \mu$, in (3.6) and (3.7) can be chosen so that $\pi'_\nu = b_1'' r$, for some $r \in K$ of value $v(r) = 0$. Note also that U'_ν/K is a Kummer extension, $[U'_\nu : K] = p$ and U'_ν is inertial over K . This implies that U'_ν is generated over K by a p -th root of an element $u'_\nu \in K$, which can be chosen so that $v(u'_\nu) = 0$ and $A_\varepsilon(u'_\nu, \pi'_\nu; K) \cong V'_\nu$. It is now easily deduced from the skew-symmetry and the \mathbb{Z} -bilinearity of symbols that

$$\begin{aligned} V'_\nu \otimes_K A_\varepsilon(a''_1, b''_1; K) &\cong (U'_\nu/K, \sigma'_\nu, r) \otimes_K A_\varepsilon(u'_\nu, b''_1; K) \otimes_K A_\varepsilon(a''_1, b''_1; K) \\ &\cong (U'_\nu/K, \sigma'_\nu, r) \otimes_K A_\varepsilon(u'_\nu a''_1, b''_1; K). \end{aligned}$$

Since, however, $[(U'_\nu/K, \sigma'_\nu, r)] \in \text{IBr}(K)$, it is clear from (3.1) (ii) that the obtained result contradicts the minimum condition on $\mu + \nu$. This contradiction proves that the system Ψ has the claimed property, which implies in

conjunction with (2.2) and [37], Theorem 1, that $V \otimes_K T \in d(K)$. In view of Kummer theory, the obtained results lead to the following conclusion:

(3.8) $\text{ind}(V \otimes_K T) = p^{\mu+\nu}$, $\nu \leq m_p$, $\nu + 2\mu \leq \tau(p)$, and $V \otimes_K T$ possesses a maximal subfield Θ which is a totally ramified abelian extension of K with $\mathcal{G}(\Theta/K)$ of exponent p .

Observing now that $\mu \leq [(\tau(p) - \nu)/2]$, one proves that $\mu + \nu \leq [(\tau(p) + \nu)/2] \leq \beta'_p$, which yields (3.4) in the case of $m = 1$. Suppose now that $m \geq 2$ and take an algebra $D_1 \in d(K)$ so that $[D_1] = p^{m-1}[D]$. Applying (3.8) to D_1 , one concludes that there exists a totally ramified abelian extension Θ_1 of K in $K(p)$ of degree dividing $p^{\beta'_p}$, such that $[D_1 \otimes_K \Theta_1] \in \text{IBr}(\Theta_1)$. Since $p^{m-1}[D \otimes_K \Theta_1] = [D_1 \otimes_K \Theta_1]$ and \widehat{K} coincides with the residue field of (Θ_1, v_{Θ_1}) , it is now easy to obtain step-by-step from (3.2) that $[D \otimes_K \Theta_m] \in \text{IBr}(\Theta_m)$, for some totally ramified extension Θ_m/K , chosen so that $\Theta_m \subseteq K(p)$ and $[\Theta_m : K] \mid p^{m\beta'_p}$. This, combined with the fact that $\text{ind}(D) \mid [\Theta_m : K]$, completes the proof of (3.4).

It remains to be seen that $\text{Brd}_p(K) \geq \beta'_p$. It is easy to see that, for each pair (μ', ν') of integers with $0 \leq \nu' \leq m_p$ and $0 \leq \mu' \leq [(\tau(p) - \nu)/2]$, there exist K -algebras D' , V' and T' , such that $D' \cong V' \otimes_K T'$, V'/K is NSR, T'/K is totally ramified, $\text{ind}(V') = p^{\nu'}$, $\text{ind}(T') = p^{\mu'}$, and $[D']$, $[V']$ and $[T']$ are elements of ${}_p\text{Br}(K)$. The obtained results indicate that $\text{Brd}_p(K) \geq [(\tau(p) + m_p)/2]$, which completes the proof of Theorem 1.3.

Corollary 3.4. *Let (K, v) be a Henselian field with $\text{Brd}_p(\widehat{K}) < \infty$ and $\text{Brd}_p(K) = \infty$, for some $p \neq \text{char}(\widehat{K})$. Then the following alternative holds:*

(i) *For each pair (n, k) of positive integers with $n \leq k$, there exists $D_{n,k} \in d(K)$, such that $\exp(D_{n,k}) = p^n$ and $\text{ind}(D_{n,k}) = p^k$;*

(ii) *$p = 2$, the group $\text{Br}(K)_2$ has exponent 2, and $d(K)$ contains algebras D_n , $n \in \mathbb{N}$, with $\text{ind}(D_n) = 2^n$.*

The latter occurs if and only if \widehat{K} is Pythagorean.

Proof. Theorem 1.3 and our assumptions show that $\tau(p) = \infty$, and in case $\varepsilon_p \notin \widehat{K}$, we have $r_p(\widehat{K}) = \infty$. In view of [52], Theorem 2, and [28], Theorem 3.16, this implies that if $p > 2$ or \widehat{K} is not Pythagorean, then $K(p)$ contains as a subfield a \mathbb{Z}_p -extension Γ_p of K . Furthermore, it follows from the Henselian property of (K, v) (cf. [23], page 135) that Γ_p can be chosen among the subfields of K_{ur} . These observations enable one to deduce Corollary 3.4 (i) and the concluding part of Corollary 3.4 (ii) from [37], Theorem 1, and Lemmas 3.1 and 3.2. Suppose now that \widehat{K} is Pythagorean and $p = 2$. By [28], Theorem 3.16, then K is also Pythagorean, so it follows from [52], Theorem 2, that K does not possess quartic cyclic extensions. The obtained result implies together with [36], (16.6), that $\text{Br}(E)_2$ does not contain an element of order 2, and so completes the proof of Corollary 3.4. \square

Remark 3.5. Let (K, v) be a virtually perfect Henselian field with $\text{char}(K) = q > 0$, $[K : K^q] = q^{n(q)}$ and the properties required by Lemma 2.1. Using Remark 3.3 and arguing as in the proof of Theorem 1.3, one obtains that

$\text{Brd}_q(K) \geq [n(q)/2]$. Also, it follows from (1.1) (ii), Albert's theorem (cf. [1], Ch. VII, Theorem 28) and [7], Lemma 3.5, that $\text{Brd}_q(K) \leq n(q)$. In Section 5 we find $\text{Brd}_q(K)$ when \widehat{K} is perfect and (K, v) is maximally complete.

4. Computations of Brauer and absolute Brauer p -dimensions, and description of index-exponent relations

In this Section we use Theorem 1.3 for computing $\text{Brd}_p(K)$ in several interesting special cases, beginning with the following result:

Theorem 4.1. *In the setting of Theorem 1.3, let \widehat{K} be a p -quasilocal nonreal field and $m_p > 0$. Then:*

- (i) $\text{Brd}_p(K) = \infty$ if and only if $m_p = \infty$ or $\tau(p) = \infty$ and $\varepsilon_p \in \widehat{K}$;
- (ii) When $\text{Brd}_p(K) < \infty$, it satisfies the equality $\text{Brd}_p(K) = u_p$, where $u_p = [(\tau(p) + m_p)/2]$, if $\varepsilon_p \in \widehat{K}$; $u_p = m_p$, otherwise.

Proof. Theorem 4.1 (i) is obviously contained in Theorem 1.3, so we have to prove Theorem 4.1 (ii). Our argument relies on the following facts:

(4.1) (i) $\text{ind}(I) = \exp(I)$, provided that $I \in d(K)$ is inertial and $[I] \in \text{Br}(K)_p$; in this case, $[I] \in \text{Br}(U/K)$, for a given inertial extension U of K in $K(p)$ if and only if $\text{ind}(U) \mid [U : K]$; when $[U : K] \mid \text{ind}(I)$, U is embeddable in I as a K -subalgebra;

(ii) $\text{ind}(J \otimes_K J')$ equals $\text{ind}(J)$ or $\text{ind}(J')$, provided that $J, J' \in d(K)$, J/K is inertial, J'/K is NSR, and $\text{Br}(K)_p$ contains $[J]$ and $[J']$.

Statements (4.1) can be deduced from (3.1), [6], Theorems 3.1 and 4.1, and [23], Theorem 5.15. They imply in conjunction with Lemma 3.1 that $\text{ind}(W) \mid \exp(W)^{m_p}$, for each $W \in d(K)$ inertially split over K . At the same time, it follows from Lemma 3.1 and [37], Theorem 1, that there is an NSR-algebra $W' \in d(K)$ with $\text{ind}(W') = p^{m_p}$ and $\exp(W') = p$. Observe now that, by (3.1) (iii), $d(K)$ consists of inertially split K -algebras in case $\varepsilon_p \notin \widehat{K}$ or $\tau(p) = 1$. In particular, this yields $\text{Brd}_p(K) = m_p$, so it remains for us to prove Theorem 4.1 (ii), under the hypothesis that $\varepsilon_p \in \widehat{K}$ and $\tau(p) \geq 2$. In view of Theorem 1.3, one may consider only the case where $\text{Br}(\widehat{K})_p \neq \{0\}$. It is easily obtained from [37], Theorem 1, and Lemmas 3.1 and 3.2 that there exists $\Delta \in d(K)$ with $\exp(\Delta) = p$ and $\text{ind}(\Delta) = p^{\mu(p)}$, where $\mu(p) = [(m_p + \tau(p))/2]$. This implies $\text{Brd}_p(K) \geq \mu(p)$, so our objective is to prove that $\text{Brd}_p(K) \leq \mu(p)$. Fix an algebra $D \in d(K)$ so that $\exp(D) = p^n$, for some $n \in \mathbb{N}$, and take S, V and T as in (3.1) (i). We prove that $\text{ind}(D) \mid p^{n\mu(p)}$, proceeding by induction on n . It follows from (4.1) (ii) that $\text{ind}(S \otimes_K V)$ equals $\text{ind}(S)$ or $\text{ind}(V)$. Since $\text{ind}(D) \mid \text{ind}(S \otimes_K V)\text{ind}(T)$, this means that it is sufficient for the proof of Theorem 4.1 (ii) to show that $\text{ind}(S)\text{ind}(T)$ and $\text{ind}(V)\text{ind}(T)$ divide $p^{n\mu(p)}$. The assumptions that \widehat{K} is nonreal, $\text{Br}(\widehat{K})_p \neq \{0\}$ and $\varepsilon_p \in \widehat{K}$, combined with [36], (16.1), and [38], Sect. 15.1, Proposition b, ensure that $r_p(\widehat{K}) \geq 2$ (and $m_p \geq 2$ because $\tau(p) \geq 2$). Applying now (3.3) (ii), one concludes that $\text{ind}(S) \mid p^n$ and $\text{ind}(S)\text{ind}(T) \mid p^{n(1+[\tau(p)/2])} \mid p^{n\mu(p)}$. It remains to be seen

that $\text{ind}(V)\text{ind}(T) \mid p^{n\mu(p)}$. Let $\Lambda \in d(K)$ be an algebra Brauer equivalent to $V \otimes_K T$. As a step in our considerations, we prove the following assertion:

(4.2) If \widehat{K} contains a primitive p^n -th root of unity, then there exists an abelian extension Λ of K , such that $[\Lambda: K]$ is divisible by $\text{ind}(D_1)^2$ and $\mathcal{G}(\Lambda/K)$ has exponent dividing p^n .

Clearly, Kummer theory ensures the existence of an abelian and inertial extension Λ_1/K , such that $[\Lambda_1: K] = p^{nm_p}$ and $\mathcal{G}(\Lambda_1/K)$ has exponent p^n . Also, it follows from Kummer theory and the Platonov-Yanchevskij theorem [38], (3.19) (see also [23], Corollary 6.10, that there exists an abelian and totally ramified extension Λ_2/K with $\mathcal{G}(\Lambda_2/K) \cong v(D_1)/v(K)$. Hence, $[\Lambda_2: K] = e(D_1/K)$ and $\mathcal{G}(\Lambda_2/K)$ has exponent dividing $\exp(D_1)$. Identifying Λ_1 and Λ_2 with their K -isomorphic copies in K_{sep} , put $\Lambda = \Lambda_1\Lambda_2$. We prove that Λ/K has the properties required by (4.2). In the first place, it is clear from Galois theory, the assumptions on Λ_1 and the noted properties of Λ_2 that Λ/K is abelian and $\mathcal{G}(\Lambda/K)$ has exponent p^n . Secondly, it is easily obtained from (2.1) that $[\Lambda: K] = [\Lambda_1: K][\Lambda_2: K]$, and it follows from [23], Theorem 6.3, that $e(D_1/K)$ is divisible by $e(V/K) = \text{ind}(V)$ and $e(T/K) = \text{ind}(T)^2$. At the same time, the inequality $p \neq \text{char}(\widehat{K})$ enables one to deduce from Galois theory and the transitivity of ramification indices in towers of finite extensions that the fields from Λ_2/K are uniquely determined by their value groups. Since, by [23], Theorem 6.3 (and the definition of D_1), $v(T) \subseteq v(D_1)$, this implies the existence of a field $\Lambda_0 \in I(\Lambda_2/K)$ with $v(\Lambda_0) = v(T)$. We show that $[D_1] \in \text{Br}(\Lambda_0\Lambda_1/K) \cup \text{Br}(\Lambda_2/K)$. Let $D' \in d(\Lambda_0)$ be a representative of the class $[D_1 \otimes_K \Lambda_0]$. It follows from Lemmas 3.1, 3.2 and the choice of Λ_0 that D'/Λ_0 is inertially split. Hence, by (3.1) and (3.3) (i), $[D'] = [S' \otimes_{\Lambda_0} V']$ (in $\text{Br}(\Lambda_0)$), where $S', V' \in d(\Lambda_0)$, S'/Λ_0 is inertial, and V'/Λ_0 is NSR. Observing that $\exp(D') \mid p^n$, and applying (4.1) (i) and [23], Corollary 5.13 and Theorem 5.15, one obtains the following:

$$(4.3) \quad \text{ind}(S') \mid p^n, \exp(V') \mid p^n \text{ and } \text{ind}(V') = e(D_1/K)/\text{ind}(T)^2.$$

Note also that $\text{ind}(D')$ equals $\text{ind}(S')$ or $\text{ind}(V')$ (by (4.1) (iii)). At the same time, since $\Lambda_1\Lambda_0/\Lambda_0$ is inertial and Λ_2/Λ_0 is totally ramified, it is easy to see that $[\Lambda_1\Lambda_2: \Lambda_0] = p^{m_p n} \text{ind}(V')$. In addition, it follows from Lemma 3.1 (applied to V' over Λ_0) and the inequality $m_p \geq 2$, that $[\Lambda_1\Lambda_0: \Lambda_0] = p^{nm_p}$ is divisible by $\text{ind}(V')$ and by p^{2n} . Summing-up the obtained results, one concludes that $[\Lambda: \Lambda_0]$ is divisible by $\text{ind}(D')^2$ and $[\Lambda: K]$ is divisible by $\text{ind}(V)^2 \text{ind}(T)^2$. This, combined with the fact that $\text{ind}(D_1) \mid \text{ind}(V)\text{ind}(T)$, proves (4.2). In view of Kummer theory, (4.2) indicates that $[\Lambda: K] \mid p^{m_p + \tau(p)}$. It is now easy to see that $\text{ind}(D_1)$ and $\text{ind}(D)$ divide $p^{n\mu(p)}$. Thus the assertion of Theorem 4.1 (ii) is proved in the special case where \widehat{K} contains a primitive p^t -th root of unity, for each $t \in \mathbb{N}$.

In order to complete the proof of Theorem 4.1 it remains to be seen that $\text{ind}(D_1) \mid p^{n\mu(p)}$, under the hypothesis that \widehat{K} does not contain a primitive p^n -th root of unity. Denote by ν the maximal integer for which \widehat{K} contains a primitive p^ν -th root of unity, and fix a representative $\Delta_\nu \in d(K)$ of the class $p^\nu[D_1] \in \text{Br}(K)$. Then it follows from (3.1) (iii), [38], Sect. 15.1, Corollary b,

and the choice of D_1 that $\exp(\Delta_\nu) \mid p^{n-\nu}$ and Δ_ν is an NSR-algebra over K . In view of Lemma 3.1, this leads to the following conclusion:

(4.4) $\text{ind}(\Delta_\nu) \mid p^{(n-\nu)m_p}$ and Δ_ν contains as a maximal subfield a totally ramified extension Φ_ν of K .

Thus it turns out that $[\Delta_\nu] \in \text{Br}(\Phi'_\nu/K)$, for some totally ramified extension Φ'_ν of K in $K(p)$ of degree $[\Phi'_\nu:K] = \text{ind}(\Delta_\nu)$. This implies $\exp(D_1 \otimes_K \Phi'_\nu) \mid p^\nu$. Taking now into account that $\widehat{\Phi}'_\nu = \widehat{K}$ (whence $r_p(\widehat{\Phi}'_\nu) = r_p(\widehat{K})$), one obtains from (2.3) and the already proved part of Theorem 4.1 (ii) that $\text{ind}(D_1 \otimes_K \Phi'_\nu) \mid p^{\nu\mu(p)}$. It is now easy to see that $\text{ind}(D_1) \mid p^{nm_p} p^{\nu[(\tau(p)-m_p)/2]}$, which completes our proof. \square

When (K, v) and p satisfy the conditions of Theorem 1.3, and $\text{Brd}_p(K) < \infty$, the description of index-exponent relations in $\text{Br}(K)_p$ does not depend only on $\text{Brd}_p(K)$. Here we illustrate this in the case where \widehat{K} is a local field.

Corollary 4.2. *Under the hypotheses of Theorem 1.3, suppose that $\text{Brd}_p(K) < \infty$, \widehat{K} is a local field, and in case $\varepsilon_p \in \widehat{K}$, put $r'_p(\widehat{K}) = r_p(\widehat{K}) - 1$ and $m'_p = \min\{\tau(p), r'_p(\widehat{K})\}$. For each $n \in \mathbb{N}$, put $\mu(p, n) = nm_p$, if $\varepsilon_p \notin \widehat{K}$, and $\mu(p, n) = nm'_p + \nu(m_p - m'_p + [\tau(p) - m_p]/2)$, provided that $\varepsilon_p \in \widehat{K}$ and ν is the greatest integer for which \widehat{K} contains a primitive p^ν -th root of unity. Then $d(K)$ contains an algebra $D_{n,k}$ with $\exp(D_{n,k}) = p^n$ and $\text{ind}(D_{n,k}) = p^k$ if and only if k is an integer satisfying the conditions $n \leq k \leq \mu(p, n)$.*

Proof. By assumption, \widehat{K} is endowed with a discrete valuation ω , such that the residue field \widehat{K}_ω of (\widehat{K}, ω) is finite and \widehat{K} is complete with respect to the topology induced by ω . Denote by $|\widehat{K}_\omega^*|$ the order of \widehat{K}_ω^* , put $r = r_p(\widehat{K})$ and let $\widehat{K}(p)_{\text{ab}}$ be the compositum of all abelian finite extensions of \widehat{K} in $\widehat{K}(p)$. Suppose first that $\varepsilon_p \notin \widehat{K}$. Then it is well-known that

- (4.5) (i) $\mathcal{G}(\widehat{K}(p)/\widehat{K}) \cong \mathbb{Z}_p$, if $p \neq \text{char}(\widehat{K}_\omega)$;
(ii) $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ is a free pro- p -group [45], whence $\mathcal{G}(\widehat{K}(p)_{\text{ab}}/\widehat{K}) \cong \mathbb{Z}_p^r$, provided that $\text{char}(\widehat{K}) = 0$ and $\text{char}(\widehat{K}_\omega) = p$; in this case, \widehat{K} is a finite extension of the field \mathbb{Q}_p of p -adic numbers, and $r = [\widehat{K}:\mathbb{Q}_p] + 1$ (see also [44], Ch. II, Theorem 3).

Note also that, in this case, $d(K)$ consists of inertially split K -algebras. These observations enable one to deduce the assertion of Corollary 4.2 from (4.1), Lemma 3.1 and [37], Theorem 1.

Henceforth, we assume that $\varepsilon_p \in \widehat{K}$. It is easily obtained from Theorem 4.1 (ii) that $\text{Brd}_p(K) \leq \mu(p, n)$, for any $n \in \mathbb{N}$, $n \leq \nu$. In view of (3.1) and (4.1), this enables one to deduce from [37], Theorem 1, and Lemmas 3.1 and 3.2 that there exist algebras $D_{n,k} \in d(K)$, $k = n, \dots, \mu(p, n)$, such that $\exp(D_{n,k}) = p^n$ and $\text{ind}(D_{n,k}) = p^k$, for each index k . The proof of Corollary 4.2 in the remaining case where $n > \nu$ relies on the following results (cf. [44], Ch. II, Theorem 4):

(4.6) (i) $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ is a Demushkin group and $\mathcal{G}(\widehat{K}(p)_{\text{ab}}/\widehat{K}) \cong \mathbb{Z}_p^{r-1} \times \mathbb{Z}/p^\nu \mathbb{Z}$;

(ii) $r = 2$, if $p \neq \text{char}(\widehat{K}_\omega)$; if $p = \text{char}(\widehat{K}_\omega)$ and $\text{char}(\widehat{K}) = 0$, then \widehat{K}/\mathbb{Q}_p is a finite extension and $r = [\widehat{K} : \mathbb{Q}_p] + 2$.

This means that $K(p) \cap K_{\text{ur}}$ is a Galois extension of K with a Galois group isomorphic to $\mathbb{Z}_p^{r-1} \times \mathbb{Z}/p^\nu \mathbb{Z}$ (see, e.g., [23], page 135, or [10], (2.3)). Let now $\Delta_\nu \in d(K)$ be an algebra defined in accordance with (4.4), and let U_ν be an inertial lift over K of $\widehat{\Delta}_\nu$. Then $[\Delta_\nu] \in p^\nu \text{Br}(K)$ and Δ_ν is NSR over K , so it follows from Galois theory, Lemma 3.1 and [38], Sect. 15.1, Corollary b, that U_ν/K is a Galois extension, such that $\mathcal{G}(U_\nu/K)$ is a homomorphic image of $\mathbb{Z}_p^{m'_p}$. Since $[U_\nu : K] = \text{ind}(\Delta_\nu)$, this ensures that if $\exp(\Delta_\nu) = p^{n'}$, then $\text{ind}(\Delta_\nu) \mid p^{n'm'_p}$. The obtained result is a refinement of (4.4), which enables one to show (as in the final part of the proof of Theorem 4.1 (ii)) that $\text{ind}(D) \mid p^{\mu(p,n)}$ whenever $n \in \mathbb{N}$, $D \in d(K)$ and $\exp(D) \mid p^n$. Note finally that Lemmas 3.1, 3.2 and [37], Theorem 1, imply for each $n \in \mathbb{N}$, the existence of algebras $D_{n,k} \in d(K)$, $k = n, \dots, \mu(p, n)$, with $\exp(D_{n,k}) = p^n$ and $\text{ind}(D_{n,k}) = p^k$, for each index k , so Corollary 4.2 is proved. \square

Retaining assumptions and notation as in Corollary 4.2 with its proof, and using (2.3), (4.5) and (4.6) together with Galois theory and [27], Proposition 5.4, one obtains the following result:

(4.7) $\text{abrd}_p(K) = 1 + [\tau(p)/2]$, provided that $p \neq \text{char}(\widehat{K}_\omega)$; $\text{abrd}_p(K) = \tau(p)$, if $\text{char}(\widehat{K}) = 0$ and $\text{char}(\widehat{K}_\omega) = p$.

Our next result is an immediate consequence both of Theorems 1.3 and 4.1. It plays a role in the proof of (1.3).

Corollary 4.3. *In the setting of Theorem 1.3, let $\text{Brd}_p(\widehat{K}) = 0$. Then:*

- (i) $\text{Brd}_p(K) = \infty$ if and only if $m_p = \infty$ or $\tau(p) = \infty$ and $\varepsilon_p \in \widehat{K}$;
- (ii) When $\text{Brd}_p(K) < \infty$, it satisfies the equality $\text{Brd}_p(K) = u_p$, where $u_p = [(\tau(p) + m_p)/2]$, if $\varepsilon_p \in \widehat{K}$; $u_p = m_p$, otherwise.

Corollary 4.4. *Assume that (K, v) is a Henselian field with $\text{Brd}_2(K) < \infty$, \widehat{K} formally real and $\text{Br}(\widehat{K}(\sqrt{-1}))_2 = \{0\}$, and in the setting of Theorem 1.3, put $m'_2 = \tau(2)$ in case $r_2(\widehat{K}) > \tau(2)$, and $m'_2 = \min(r_2(\widehat{K}) - 1, \tau(2))$, otherwise. Then $\text{Brd}_2(K) = 1 + [(m'_2 + \tau(2))/2]$.*

Proof. It follows from (1.1) (ii) and the triviality of $\text{Br}(\widehat{K}(\sqrt{-1}))_2$ that $\text{Br}(\widehat{K})_2 = \text{Br}(\widehat{K}(\sqrt{-1})/\widehat{K})$. As \widehat{K} is formally real, this implies that $\text{Br}(\widehat{K})_2 \neq \{0\}$ and $\text{ind}(\widehat{D}) = 2$, for every $\widehat{D} \in d(\widehat{K})$ representing a nonzero element of $\text{Br}(\widehat{K})_2$. Applying (3.2) and arguing as in the proof of (3.8) and (3.4), one concludes that if $D \in d(K)$ and $\exp(D) = 2^m$, for some $m \in \mathbb{N}$, then there exists a totally ramified extension Θ/K , such that $D \otimes_K \Theta \in \text{IBr}(\Theta)$ and $[\Theta : K] = 2^{m \cdot [(m'_2 + \tau(2))/2]}$. These observations indicate that $\text{ind}(D) \mid 2 \cdot 2^{m \cdot [(m'_2 + \tau(2))/2]} = 2^{1+m \cdot [(m'_2 + \tau(2))/2]}$ and so prove that $\text{Brd}_2(K) \leq 1 + [(m'_2 + \tau(2))/2]$. On the other hand, one deduces from [37], Theorem 1,

and Lemmas 3.1 and 3.2 (as in the proof of (3.8)) that $d(K)$ contains an algebra Δ with $\exp(\Delta) = 2$ and $\text{ind}(\Delta) = 2^{1+[(m'_2+\tau(2))/2]}$, which completes the proof of Corollary 4.4. \square

Corollary 4.5. *Let C be an algebraically closed field and $C_n = C((X_1)) \dots ((X_n))$ an iterated formal Laurent power series field in n indeterminates over C , where $n \in \mathbb{N}$. Then $\text{Brd}_p(C_n) = \text{abrd}_p(C_n) = [n/2]$, for every $p \in \mathbb{P}$ different from $\text{char}(C)$. In addition, for each pair (ν, κ) of positive integers satisfying the inequalities $\nu \leq \kappa \leq \nu[n/2]$, there exists $T_{\nu, \kappa} \in d(C)$ with $\exp(T_{\nu, \kappa}) = p^\nu$ and $\text{ind}(T_{\nu, \kappa}) = p^\kappa$.*

Proof. Let v_n be the standard \mathbb{Z}^n -valued valuation of C_n with respect to the inverse-lexicographic ordering on \mathbb{Z}_n . It is well-known (Krull's theorem, see [15], Sects 4.2 and 18.4) that (C_n, v_n) is maximally complete, whence Henselian, and C is its residue field. As C is algebraically closed, this implies $\bar{F} \cong C$ and $v_n(F) \cong \mathbb{Z}^n$, for each finite extension F/C_n . Observing also that $r_p(C) = \text{Brd}_p(C) = 0$ and the group $\mathbb{Z}^n/p\mathbb{Z}^n$ is of order p^n when p runs across \mathbb{P} , one deduces the equalities $\text{Brd}_p(C) = \text{abrd}_p(C) = [n/2]$, $p \in \mathbb{P}$, from Corollary 4.3. Suppose finally that A_p is a finite abelian group of exponent p^ν , which is generated by at most $[n/2]$ elements, and denote by $o(A_p)$ its order. Then it follows from Kummer theory, [37], Theorem 1, and the structure of totally ramified central division algebras (cf.) that there exists $T_p \in d(C_n)$ with $v(T_p)/v(C_n) \cong A_p \times A_p$. This implies that $\exp(T_p) = p^\nu$ and $\text{ind}(T_p) = o(A_p)$. Observing finally that A_p can be chosen so that $o(A_p) = p^\kappa$ if and only if $\nu \leq \kappa \leq \nu[n/2]$, and taking into consideration that every $D \in d(C_n)$ is totally ramified over C_n relative to v_n , one completes the proof of Corollary 4.5. \square

Corollary 4.6. *Let E_0 be a real closed field and $E_n = E_0((X_1)) \dots ((X_n))$, for some $n \in \mathbb{N}$. Then $\text{Br}(E_n) = \text{Br}(E_n)_2$, $\text{Brd}_2(E_n) = \text{abrd}_2(E_n) = 1 + [n/2]$ and $\text{abrd}_p(E_n) = [n/2]$, for every $p \in \mathbb{P} \setminus \{2\}$.*

Proof. The standard \mathbb{Z}^n -valued height n valuation of E_n is Henselian with a residue field E_0 , and by the Artin-Schreier theory (cf. [29], Ch. XI, Sect. 2), $E_{0, \text{sep}} = E_0(\sqrt{-1})$. This, combined with Corollaries 4.3 and 4.5, proves that $\text{Brd}_2(E_n) = 1 + [n/2]$. Note also that a finite extension E'_n of E_n is isomorphic to E_n or to $E_0(\sqrt{-1})((X_1)) \dots ((X_n))$ depending on whether it is formally real. These observations prove that $\text{abrd}_2(E_n) = 1 + [n/2]$. The proof of the equality $\text{Br}(E_n) = \text{Br}(E_n)_2$ relies on the fact that E_0 is an ordered field, whence it does not contain a primitive n -th root of unity, for any integer $n \geq 3$. In view of (3.1) (iii) and (3.3) (i), this proves that if $D \in d(F)$ and $2 \nmid \text{ind}(D)$, then D is inertially split. As $E_0(\sqrt{-1}) = E_{0, \text{sep}}$, one also sees that E_n does not possess inertial extensions of odd degrees (relative to v_n), and we have $\text{Br}(E_0) = \text{Br}(E_0)_2$. It can now be easily deduced from (3.1) and Lemma 3.1 that $\text{Br}(E_n)_p = \{0\}$, for every $p \in \mathbb{P} \setminus \{2\}$, which proves the assertion that $\text{Br}(E_n) = \text{Br}(E_n)_2$. As to the equalities $\text{abrd}_p(E_n) = [n/2]$, $p \in \mathbb{P} \setminus \{2\}$, they follow from the obtained result, the

equality $E_{0,\text{sep}} = E_0(\sqrt{-1})$, Corollary 4.5 and the noted structural property of finite extensions of E_n , so Corollary 4.6 is proved. \square

The fulfillment of the conditions of Corollary 4.6 guarantees that E_n is Pythagorean (cf. [28], Theorem 3.16), so it follows from [36], (16.6), that $\text{Br}(E_n)$ is a group of exponent 2. It is therefore easily obtained from [37], Theorem 1, and Lemma 3.2 that there are algebras $D_k \in d(E_n)$, $k = 1, \dots, 1 + [n/2]$, with $\text{ind}(D_k) = 2^k$, for any index k .

Remark 4.7. Under the hypotheses of Corollary 4.5, put $\text{char}(C) = q \geq 0$. When $q > 0$, the proof of Corollary 4.5 enables one to deduce from [2], Theorem 3.3, that $\text{Brd}_q(C_n) = \text{Brd}(C_n) = \text{abrd}(C_n) = n-1$ (see Proposition 5.1). On the other hand, if $q = 0$, then Corollary 4.5 yields $\text{Brd}(C_n) = \text{abrd}(C_n) = [n/2]$. Note also that $\text{Brd}_p(F) = \text{abrd}_p(F) = n$, $p \in \mathbb{P} \setminus \{q\}$, for every finitely-generated field extension F/C_n of transcendency degree 1. Indeed, by [32], Corollary 1.4, $\text{abrd}_p(F) \leq n$, $p \in \mathbb{P} \setminus \{q\}$, and the inequalities $\text{Brd}_p(F) \geq n$, $p \in \mathbb{P} \setminus \{q\}$, are obtained by analyzing the proofs of [7], Theorem 1.2 (ii), and of Krashen's results on page 37 of [32]. When $q = 0$, these facts make interest from the perspective of the Standard Conjecture, as stated in [32], for fields E with $\text{abrd}(E) < \infty$ (see [7], Remark 5.5).

Corollary 4.5 and our next result singles out several frequently used types of Henselian fields (K, v) , which have the property that $\text{Brd}_p(K)$ fully determines index-exponent relations in $\text{Br}(K)_p$ whenever $p \in \mathbb{P}$ and $p \neq \text{char}(\hat{K})$.

Proposition 4.8. *With assumptions being as in Theorem 1.3, let \hat{K} be a field of one of the following three types: (i) a global field; (ii) the function field of an algebraic surface defined over an algebraically closed field; (iii) the function field of an algebraic curve defined over a perfect PAC-field E_0 with $\text{cd}_p(\mathcal{G}_{E_0}) \neq 0$. Then $\text{Brd}_p(K) = \infty$, if $\tau(p) = \infty$, and $\text{Brd}_p(K) = \text{abrd}_p(K) = 1 + \tau(p)$ when $\tau(p) < \infty$. Moreover, if $\tau(p) < \infty$, then for each $n \in \mathbb{N}$, there are algebras $D_{n,k} \in d(K)$, $k = n, \dots, n(1 + \tau(p))$, such that $\exp(D_{n,k}) = p^n$ and $\text{ind}(D_{n,k}) = p^k$, for each index k .*

Proof. The assertions about $\text{abrd}_p(K)$ follow at once from (2.1), (2.3), the assertions concerning $\text{Brd}_p(K)$ and the fact that the type of \hat{K} is preserved by its finite extensions. Therefore, it suffices to determine $\text{Brd}_p(K)$ and to prove the concluding statement of Proposition 4.8. Our assumptions show that \hat{K} has nonequivalent discrete valuations \hat{v}_n , $n \in \mathbb{N}$, satisfying the following conditions, for each index n (see [4], Ch. II, Sects. 3 and 10):

(4.8) (i) $r_p(\hat{K}_n) > 0$ and \hat{K}_n contains a primitive p^n -th root of unity, where \hat{K}_n is the residue field of (\hat{K}, \hat{v}_n) .

Statements (4.8) enable one to deduce the former part of the following assertions from Grunwald-Wang's theorem, Kummer theory and [52], Theorem 2:

(4.9) (i) $d(\hat{K})$ contains algebras $\tilde{\Delta}_n$, $n \in \mathbb{N}$, such that $\tilde{\Delta}_n \otimes_{\hat{K}} \hat{K}_{\hat{v}_n} \in d(\hat{K}_{\hat{v}_n})$, $\text{ind}(\tilde{\Delta}_n \otimes_{\hat{K}} \hat{K}_{\hat{v}_n}) = \exp(\tilde{\Delta}_n \otimes_{\hat{K}} \hat{K}_{\hat{v}_n}) = p^n$ and $\tilde{\Delta}_n \otimes_{\hat{K}} \hat{K}_{\hat{v}_n}$ is NSR over $\hat{K}_{\hat{v}_n}$, for each n ;

(ii) With notation being as in (i), every finite abelian group G_n of exponent $e(G_n)$ dividing p^n is isomorphic to $\mathcal{G}(\tilde{K}'_n/\tilde{K})$, for some Galois extension \tilde{K}'_n of \tilde{K} in $\tilde{K}(p)$, which can be chosen so that $\tilde{\Delta}_n \otimes_{\tilde{K}} \tilde{K}'_n \in d(\tilde{K}'_n)$.

In addition, it follows from (4.8) (i) and Grunwald-Wang's theorem that, for each $n, m \in \mathbb{N}$, there exists cyclic extensions $\tilde{M}_{n,m}$ of \tilde{K} in $\tilde{K}(p)$, such that $[\tilde{M}_{n,m} : \tilde{K}] = p^n$ and $[\tilde{M}_{n,1} \dots \tilde{M}_{n,m} : \tilde{K}] = [\tilde{M}_{n,1} \dots \tilde{M}_{n,m} \tilde{K}_{\hat{v}_n} : \tilde{K}_{\hat{v}_n}] = p^{nm}$. In view of (2.2) and Galois theory, the obtained result proves (4.9) (ii). It is now easy to see that $r_p(\tilde{K}) = \infty$ as well as to show that if $\tau(p) = \infty$, then $\text{Brd}_p(K) = \infty$. Suppose further that $\tau(p) < \infty$. Observing that $\text{Brd}_p(\tilde{K}) = 1$, one obtains from Theorem 1.3 that $\text{Brd}_p(K) \leq 1 + \tau(p)$. It remains for us to prove the concluding assertion of Proposition 4.8. Fix positive integers n and k so that $n \leq k \leq n\tau(p)$, choose G_n to be $\tau(p)$ -generated of order $o(G_k) = p^{k-n}$ and with $e(G_k) \mid p^n$, and take $\tilde{\Delta}_n \in d(\tilde{K})$ and $\tilde{K}'_n \in I(\tilde{K}(p)/\tilde{K})$ in accordance with (4.9). Also, let $\Delta_n \in d(K)$ and $K'_n \in I(K_{\text{sep}}/K)$ be inertial lifts over K of $\tilde{\Delta}_n$ and \tilde{K}'_n , respectively. Then it follows from Lemma 3.1 and the choice of G_n that there is an NSR-algebra $V_n \in d(K)$, such that K'_n is K -isomorphic to a maximal subfield of V_n . In particular, this ensures that $\exp(V_n) = e(G_n)$ and $\text{ind}(V_n) = p^{k-n}$. The obtained results, combined with (4.9) and [23], Theorem 5.15, imply that $\Delta_n \otimes_K V_n \in d(K)$, $\exp(\Delta_n \otimes_K V_n) = p^n$ and $\text{ind}(\Delta_n \otimes_K V_n) = p^k$. Thus it turns out that $\text{Brd}_p(K) \geq 1 + \tau(p)$, so Proposition 4.8 is proved. \square

Remark 4.9. In the setting of Theorem 1.3, let \hat{K} be the function field of an algebraic curve over a field E_0 with $\text{cd}_p(\mathcal{G}_{E_0}) = 0$. This means that the degrees of finite extensions of E_0 in $E_{0,\text{sep}}$ are not divisible by p , so it follows from (1.1) (ii) and Tsen's theorem (cf. [38], Sect. 19.4) that $\text{abrd}_p(\hat{K}) = 0$. Returning to the proof of Proposition 4.8, one also sees that $r_p(\hat{K}) = \infty$. Since $p \neq \text{char}(\hat{K})$, these observations and Corollary 4.3 imply that $\text{Brd}_p(K) = \text{abrd}_p(K) = \tau(p)$, and for each pair (n, k) of positive integers with $n \leq k \leq n\tau(p)$, there is $D_{n,k} \in d(K)$, such that $\exp(D_{n,k}) = p^n$ and $\text{ind}(D_{n,k}) = p^k$.

We refer the reader to [8], for a proof of the following result (obtained by combining the method used in this Section with results of [16] and [17]).

Proposition 4.10. *Let E be a field containing a primitive p -th root of unity ε , for some $p \in \mathbb{P}$, and let $r_p(E) \geq 2$ and $\mathcal{G}(E(p)/E)$ be metabelian. Then:*

- (i) $r_p(E') = r_p(E)$, for every finite extension E' of E in $E(p)$ unless E is formally real, $p = 2$ and $r_2(E) < \infty$;
- (ii) $\text{Brd}_p(E) = \infty$, provided that $r_p(E) = \infty$; $\text{Brd}_p(E) = [r_p(E)/2]$ in case E is nonreal and $r_p(E) < \infty$;
- (iii) If E is formally real, then $p = 2$ and E is Pythagorean; in particular, $\text{Br}(E)_2$ is a group of exponent 2;
- (iv) If E is formally real and $r_2(E) < \infty$, then $\text{Brd}_2(E) = [(1 + r_2(E))/2]$, and for each finite extension E' of E in $E(2)$, $r_2(E) - 1 \leq r_2(E') \leq r_2(E)$.

5. The Brauer q -dimension of a maximally complete field of characteristic q

The purpose of this Section is to shed light on the behaviour of $\text{Brd}_q(K)$, for a maximally complete field (K, v) of characteristic $q > 0$. Our main result in this direction can be stated as follows:

Proposition 5.1. *Let (K, v) be a maximally complete field with $\text{char}(K) = q > 0$, and let $\tau(q)$ be the dimension of $v(K)/qv(K)$ over \mathbb{F}_q . Then:*

- (i) $\text{Brd}_q(K) = \infty$, provided that $\tau(q) = \infty$ or \widehat{K} is not virtually perfect;
- (ii) If \widehat{K} is perfect and $\tau(q) < \infty$, then $\tau(q) - 1 \leq \text{Brd}_q(K) \leq \tau(q)$; in order that $\text{Brd}_q(K) = \tau(q)$ it is necessary and sufficient that $r_q(\widehat{K}) \geq \tau(q)$.

Proof. Our assertions can be deduced from Remark 3.3, [7], Lemma 3.5, [2], Theorem 3.3, and the following lemma. \square

Lemma 5.2. *Let (K, v) be a Henselian field with $\text{char}(K) = q > 0$ and $v(K) \neq qv(K)$, and $\tau(q)$ be the dimension of $v(K)/qv(K)$ over \mathbb{F}_q . Then:*

- (i) For each $\pi \in K^*$ with $v(\pi) \notin qv(K)$, there exists a sequence L_m : $m \in \mathbb{N}$, of degree q cyclic extensions of K in K_{sep} , such that the compositum $M_m = L_1 \dots L_m$ is totally ramified over K , $[M_m : K] = q^m$, and $v(\pi) \in q^m v(M_m)$, for each index m ;
- (ii) There exists $T_n \in d(K)$ with $\exp(T_n) = q$ and $\text{ind}(T_n) = q^{n-1}$, n being an integer ≥ 2 , except, possibly, in the case where \widehat{K} is virtually perfect and $n > \tau(q) + \log_q[\widehat{K} : \widehat{K}^q]$.

Proof. One may assume without loss of generality that π is chosen so that $v(\pi) < 0$. For each $m \in \mathbb{N}$, put $f_m(X) = X^q - X - \pi_m$, where $\pi_m = \pi^{1+qm}$, and denote by L_m the root field of the polynomial $f_m(X)$ in K_{sep} . Also, let \mathbb{F} be the prime subfield of K , and $\rho(K) = \{u^q : u \in K\}$. It is well-known that $\rho(K)$ is a vector \mathbb{F} -subspace of K , and it follows from the Henselian property of (K, v) and the choice of π that the cosets $\pi_m + \rho(K)$, $m \in \mathbb{N}$, are linearly independent over \mathbb{F} . In view of the Artin-Schreier theorem, this implies that $f_m(X)$ is irreducible over K , L_m/K is cyclic, $[L_m : K] = q$ and $[M_m : K] = q^m$, for each $m \in \mathbb{N}$. Moreover, our argument proves that every degree q extension of K in the compositum L of the fields L_m , $m \in \mathbb{N}$, is cyclic and totally ramified over K . We show that the field extensions M_m/K are totally ramified, for all m . The preceding observations prove our assertion in the special case where v is discrete and \widehat{K} is perfect. For the proof in general, consider the valuation φ of the field $\Phi = \mathbb{F}(\pi)$ induced by v . Clearly, φ is nontrivial, which implies that it is discrete (cf. [4], Ch. II, Lemma 3.1). Observe also $(K, v)/(\Phi, \varphi)$ contains as a valued subfield a Henselization (Φ', φ') of (Φ, φ) (cf. [15], Sect. 15.3). In particular, (Φ', φ') is a Henselian discrete valued field. Let now Ψ_m be the root field in Φ'_{sep} of $f_m(X)$ (viewed as a polynomial over Φ'), for each $m \in \mathbb{N}$. Since $\varphi'(\Phi') \subseteq v(K)$, the choice of π ensures that Ψ_m/Φ' , $m \in \mathbb{N}$, are cyclic degree q extensions, $\Psi'_m = \Psi_1 \dots \Psi_m$ is an abelian extension of Φ' of degree

q^m , for each index m , and every finite extension L' of Φ' is abelian and totally ramified. Note also that $\varphi'(L')/\varphi'(\Phi')$ is a cyclic group. Identifying Φ'_{sep} with its Φ' -isomorphic copy in K_{sep} , and applying Galois theory, one concludes that $\Psi K = M$ and $\Psi \cap K = \Phi'$. Thus it turns out that the mapping of $I(\Psi/\Phi')$ into $I(M/K)$ by the rule $R' \rightarrow R'K$: $R' \in I(R'/\Phi')$, is bijective and degree-preserving. The obtained results, combined with (2.1), complete the proof of Lemma 5.2 (i).

Our objective now is to prove Lemma 5.2 (ii). Put $\pi_1 = \pi$ and suppose that there exist elements $\pi_j \in K^*$, $j = 2, \dots, n$, as well as a positive integer $\mu \leq n$, such that the cosets $v(\pi_i) + qv(K)$, $i = 1, \dots, \mu$, are linearly independent over \mathbb{F}_q , and in case $\mu < n$, $v(\pi_{\mu+1}) = \dots = v(\pi_n) = 0$, and the residue classes $\hat{\pi}_u$, $u = \mu + 1, \dots, n$, generate an extension of \hat{K}^q of degree $q^{n-\mu}$. Fix a generator λ_m of $\mathcal{G}(L_m/K)$, for each $m \in \mathbb{N}$, and for each integer ν with $2 \leq \nu \leq n$, denote by T_ν the K -algebra $\otimes_{j=2}^\nu (L_{j-1}/K, \lambda_{j-1}, \pi_j)$, where $\otimes = \otimes_K$. The definition of T_n ensures that $T_n \in S(K)$, $\exp(T_n) \mid q$ and $\deg(T_n) = q^{n-1}$, so the proof of Lemma 5.2 will be complete, if we show that $T_n \in d(K)$. Suppose first that $n = 2$. Then our assertion follows from the fact that $v(\pi_2) \notin qv(L_1)$, whereas $qv(L_1)$ contains $v(\theta)$ whenever θ lies in the norm group $N(L_1/K)$ (see also [38], Sect. 15.1, Proposition b). Henceforth, we assume that $n \geq 3$ and identify all value groups considered in the rest of the proof with their isomorphic copies in a fixed divisible hull of $v(K)$. It is easy to see that the centralizer $C_{T_n}(L_n)$ is L_n -isomorphic to $T_{n-1} \otimes_K L_n$. In view of (2.1) and Lemma 5.2 (i), this indicates that it is sufficient to establish the claimed property of T_n , under the extra hypothesis that $C_{T_n}(L_n) \in d(L_n)$. Denote by v_n the valuation of $C_{T_n}(L_n)$ extending v_{L_n} . It follows from (2.2) that $v_n(C_{T_n}(L_n))$ equals the sum of $v(M_n)$ and the group generated by the elements $q^{-1}v(\pi_{i'})$, $i' = 2, \dots, n-1$. In particular, this implies that $v(\pi_n) \notin qv_n(C_{T_n}(L_n))$. At the same time, it is easily verified that there exists a K -automorphism $\bar{\lambda}_n$ of $C_{T_n}(L_n)$ extending λ_n and acting as the identity on the canonical K -isomorphic copy of T_{n-1} in $C_{T_n}(L_n)$. Since, by the Skolem-Noether theorem (cf. [38], Sect. 12.6), $\bar{\lambda}_n$ is induced by an inner K -automorphism of T_n , we have $v_n(t_n) = v_n(\bar{\lambda}_n(t_n))$, for each $t_n \in C_{T_n}(L_n)$. This indicates that $v_n(\bar{t}_n) \in qv_n(\bar{t}_n)$, whence $\bar{t}_n \neq \pi_n$, where $\bar{t}_n = \prod_{\kappa=0}^{q-1} \bar{\lambda}_n^\kappa(t_n)$, for each $t_n \in C_{T_n}(L_n)$. The obtained result, combined with Albert's theorem (cf. [1], Ch. XI, Theorems 11 and 12), yields $T_n \in d(K)$, so Lemma 5.2 is proved. \square

Lemma 5.2 illustrates the fact that the exponent of $v(T)/v(K)$, where (K, v) is a Henselian field and $T \in d(K)$, need not divide $\exp(T)$ without the assumption that T is tame over K .

Corollary 5.3. *Let (K, v) be a maximally complete field with $\text{char}(K) = q > 0$ and $\tau(q) < \infty$, and suppose that \hat{K} is the function field of an algebraic curve defined over a perfect field \hat{K}_0 with $\text{cd}_q(\hat{K}_0) > 0$. Then $\text{Brd}_q(K) = \text{abrd}_q(K) = 1 + \tau(q)$, and for each $n \in \mathbb{N}$, there exist $D_{n,k} \in d(K)$, $k = n, \dots, n(1 + \tau(q))$, with $\exp(D_{n,k}) = q^n$ and $\text{ind}(D_{n,k}) = q^k$.*

Proof. As in the proof of Proposition 4.8, one sees that it is sufficient to prove the equality $\text{Brd}_q(K) = 1 + \tau(q)$ and our concluding assertion. Also, it follows from the condition on \widehat{K} and the results in [4], Sects. 3 and 10, that \widehat{K} has nonequivalent discrete valuations \hat{v}_t , $t \in \mathbb{N}$, trivial on \widehat{K}_0 and such that $r_q(\widehat{K}_t) > 0$, where \widehat{K}_t is the residue field of (\widehat{K}, \hat{v}_t) , for each t . It is therefore clear from Lemma 2.2 that $r_q(\widehat{K}) = \infty$. Note further that cyclic extensions of \widehat{K} in $\widehat{K}(q)$ are realizable as intermediate fields of \mathbb{Z}_q -extensions of \widehat{K} . This is obtained straightforwardly from Galois theory and Witt's lemma (cf. [11], Sect. 15, Lemma 2). These observations enable one to prove the validity of statements (4.9), for $p = q$. The obtained result makes it easy to prove the concluding assertion of Corollary by the method of proving the concluding part of Proposition 4.8. In particular, this yields $\text{Brd}_q(K) \geq 1 + \tau(q)$. On the other hand, the assumptions on \widehat{K} and K ensure that $[K : K^q] = q^{1+\tau(q)}$. Hence, by [1], Ch. VII, Theorem 28, $\text{Brd}_q(K) \leq 1 + \tau(q)$, which completes the proof of Corollary 5.3. \square

Remark 5.4. Let (K, v) be a maximally complete field with $\text{char}(K) = q \neq 0$ and $\tau(q) > 0$. Arguing as in the proof of Corollary 5.3, one obtains that $1 + \tau(q) \leq \text{Brd}_q(K) \leq 2 + \tau(q)$, provided that \widehat{K} is the function field of an algebraic surface defined over an algebraically closed field.

Theorem 1.3 and Lemma 5.2 generalize observations and calculations made in [5] in the process of characterizing the stability property in several interesting classes of Henselian fields, including those of residual characteristic zero. They can serve as a basis for finding similar characterizations of basic types of Henselian fields of any fixed finite Brauer dimension. In order to obtain sufficiently complete results, it is necessary to find an adequate generalization of the notion of a p -quasilocal field within the class of fields of Brauer p -dimensions at most equal to a given positive integer n .

6. Proofs of Theorems 1.1 and 1.2

Let (K, v) be a Henselian field with $\text{abrd}_p(\widehat{K}) < \infty$. Our first result provides lower and upper bounds for $\text{abrd}_p(K)$ in the case of $p \neq \text{char}(\widehat{K})$.

Proposition 6.1. *Let K, v and p satisfy the conditions of Theorem 1.3, and suppose that $\text{abrd}_p(\widehat{K}) < \infty$, G_p is a Sylow pro- p -subgroup of \mathcal{G}_K . Then:*

- (i) $\text{abrd}_p(K) = \infty$ if and only if $\tau(p) = \infty$;
- (ii) $\max(\text{abrd}_p(\widehat{K}) + [\tau(p)/2], \tau(p)) \leq \text{abrd}_p(K) \leq \text{abrd}_p(\widehat{K}) + \tau(p)$, provided that $\tau(p) < \infty$ and G_p is not metabelian.

Proof. Let K_p be the fixed field of G_p . Then $p \nmid [R_0 : K]$, for each finite extension R_0 of K in K_p , so it follows from (2.3) that $v(K)/pv(K) \cong v(R)/pv(R)$, for each $R \in I(K_p/K)$. At the same time, it is easy to see that $\text{Br}(R/K) \cap \text{Br}(K)_p = \{0\}$, and $\text{ind}(D_p \otimes_K R) = \text{ind}(D_p)$ and $\exp(D_p \otimes_K R) = \exp(D_p)$ whenever $D_p \in d(K)$ and $[D_p] \in \text{Br}(K)_p$. These observations show that $\text{abrd}_p(K) = \text{abrd}_p(R)$, $R \in I(K_p/K)$. Taking now

into account that K_p contains a primitive p -th root of unity, say ε_p , one deduces from Theorem 1.3 (i) that $\text{Brd}_p(K(\varepsilon_p)) = \text{abrd}_p(K(\varepsilon_p)) = \infty$ in case $\tau(p) = \infty$. In the rest of the proof we assume that $\tau(p) < \infty$. In view of (2.3) and Theorem 1.3, then $\text{abrd}_p(K) \leq \tau(p)$, which proves Proposition 6.1 (i). Denote by k_p the residue field of (K_p, v_p) , and by U_p the compositum of the inertial extensions of K_p in K_{sep} relative to v_{K_p} . It follows from Galois theory and the Henselity of (K, v) that U_p is a Galois extension of K_p with $\mathcal{G}(U_p/K_p)$ isomorphic to \mathcal{G}_{k_p} and to the Sylow pro- p -subgroups of $\mathcal{G}_{\widehat{K}}$. Thus the proof of Proposition 6.1 (ii) reduces to the special case where $K = K_p$. As G_p is not metabelian, it follows from the Mel'nikov-Tavgen' theorem and the preceding observations that $\mathcal{G}_{\widehat{K}}$ and $\mathcal{G}(U_p/K)$ are not metabelian either. Therefore, $\mathcal{G}(U_p/K_p)$ possesses a closed subgroup H_p that is a free pro- p -group of rank $r(H_p) \geq 2$. Hence, by [7], Proposition 6.1, $\text{abrd}_p(L_p) = \tau(p)$, where L_p is the fixed field of H_p . Hence, by Theorem 1.3 (ii) and the isomorphism $v(T_m)/pv(T_m) \cong v(K)/pv(K)$, we have $\text{Brd}_p(T_m) = \tau(p)$, for every $m > \tau(p)$, which completes the proof of Proposition 6.1 (i). \square

The rest of this Section is devoted to the proof of Theorems 1.1 and 1.2. Denote for brevity by Alt_∞ the group product $\prod_{n=5}^\infty \text{Alt}_n$, where Alt_n is the alternating group of degree n , for each index n . Our objective is to show that a field E with the claimed properties can be chosen so as to have a Henselian valuation v with \widehat{E} is perfect, $\text{cd}(\mathcal{G}_{\widehat{E}}) \leq 1$ and $\text{char}(\widehat{E}) = \text{char}(E)$. In order to follow a unified approach, it is convenient to put $\Pi_1(\bar{a}, \bar{b}) = \phi$ and $\Pi_0(\bar{a}, \bar{b}) = \{2\}$, for any sequence (\bar{a}, \bar{b}) satisfying the conditions of Theorem 1.1. When $\text{char}(E) = 0$, we suppose that \widehat{E} and $v(E)$ are subject to the following restrictions:

- (6.1) (i) $\Pi_0(\bar{a}, \bar{b})$ consists of those $p \in \mathbb{P}$, for which \widehat{E} contains a primitive p -th root of unity and $r_p(\widehat{E}) > 0$; $\Pi_1(\bar{a}, \bar{b})$ consists of those $\pi \in \mathbb{P}$, for which \widehat{E} contains a primitive π -th root of unity and $r_\pi(\widehat{E}) = 0$;
- (ii) For each $p_0 \in \Pi_0(\bar{a}, \bar{b})$ with $a_{p_0} = 2b_{p_0}$, $\mathcal{G}(\widehat{E}(p_0)/\widehat{E}) \cong \mathbb{Z}_{p_0}$; $r_{\pi_0}(\widehat{E}) = 2(b_{\pi_0} - [a_{\pi_0}/2])$ in case $\pi_0 \in \Pi_0(\bar{a}, \bar{b}) \setminus \Pi(\bar{a}, \bar{b})$ and $a_{\pi_0} < 2b_{\pi_0}$;
- (iii) $\tau(p) = \infty$ if and only if $a_p = \infty$; $\tau(\pi) = a_\pi$, provided that $a_\pi = 0$ or π lies in the union of $\Pi_1(\bar{a}, \bar{b})$ and $\Pi_0(\bar{a}, \bar{b}) \setminus \Pi(\bar{a}, \bar{b})$;
- (iv) $2b_\pi - 1 \leq \tau(\pi) \leq 2b_\pi$, and $\mathcal{G}(\widehat{E}(\pi)/\widehat{E})$ and the Sylow pro- π -subgroups of $\mathcal{G}_{\widehat{E}}$ are isomorphic to \mathbb{Z}_π whenever $\pi \in \Pi(\bar{a}, \bar{b}) \cap \Pi_0(\bar{a}, \bar{b})$ and $a_\pi < \infty$;
- (v) $r_p(\widehat{E}) = b_p$ and $\tau(p) = a_p$ in case $p \notin \Pi(\bar{a}, \bar{b}) \cup \Pi_0(\bar{a}, \bar{b}) \cup \Pi_1(\bar{a}, \bar{b})$.

In addition, we require that Henselian field (E, v) satisfies (6.1) holds and \widehat{E} is subject to the following restrictions:

- (6.2) (i) If $p_0 \in \Pi_0(\bar{a}, \bar{b}) \setminus \Pi(\bar{a}, \bar{b})$, $a_{p_0} = 2b_{p_0}$, and there exists $\bar{p} \notin \Pi_1(\bar{a}, \bar{b}) \cup \{p_0\}$ with $b_{\bar{p}} \neq 0$ or $a_{\bar{p}} = 0$, then \widehat{E} possesses a finite Galois extension $\widehat{E}'_{p_0} \in I(\widehat{E}_{\text{sep}}/\widehat{E})$, such that $\mathcal{G}(\widehat{E}'_{p_0}/\widehat{E})$ is an NMM-group whose Sylow p_0 -subgroup is normal and has exponent p_0 and prime index;
- (ii) $\mathcal{G}_{\widehat{E}}$ is a Frattini cover of the group $\mathbb{Z}_{p_0} \times \text{Alt}_\infty$, provided that $\Pi_0(\bar{a}, \bar{b}) = \{p_0\}$, $p_0 \notin \Pi(\bar{a}, \bar{b})$, $a_{p_0} = 2b_{p_0}$, and $b_{\bar{p}} = 0 \neq a_{\bar{p}}$ when $\bar{p} \notin \Pi_j(\bar{a}, \bar{b})$, $j = 0, 1$;

- (iii) If $\Pi_1(\bar{a}, \bar{b}) \notin \mathbb{P}$, then for each $\pi \in \Pi_1(\bar{a}, \bar{b})$, there is a finite Galois extension \widehat{E}'_π of \widehat{E} in \widehat{E}_{sep} , such that $\mathcal{G}(\widehat{E}'_\pi/\widehat{E})$ is an NMM-group with a noncyclic normal π -subgroup of exponent π and prime index;
- (iv) If $\Pi_1(\bar{a}, \bar{b}) = \mathbb{P}$ and $\widehat{\Psi}$ is the fixed field of $\Phi(\mathcal{G}_{\widehat{E}})$, then $\mathcal{G}(\widehat{\Psi}/\widehat{E}) \cong \text{Alt}_\infty$;

The latter assertion of Theorem 1.1 is proved simultaneously with the following result.

Proposition 6.2. *Let $(\bar{a}, \bar{b}) = a_p, b_p$, $p \in \mathbb{P}$, be a sequence of elements of $\mathbb{N} \cup \{0, \infty\}$, such that $a_p \geq b_p$, for each p , and let $q \in \mathbb{P}$ and $\Pi(\bar{a}, \bar{b})$, $\Pi'_1(\bar{a}, \bar{b})$ and $\Pi_q(\bar{a}, \bar{b})$ be subsets of \mathbb{P} satisfying the following conditions:*

- (i) $\Pi(\bar{a}, \bar{b}) = \{p' \in \mathbb{P} : a_{p'} = b_{p'}\}$, $\Pi_1(\bar{a}, \bar{b}) \cap (\Pi_q(\bar{a}, \bar{b}) \cup \Pi'_1(\bar{a}, \bar{b})) = \Pi'_1(\bar{a}, \bar{b}) \cap \Pi_q(\bar{a}, \bar{b}) = \emptyset$, and for each $p_1 \in \Pi_1(\bar{a}, \bar{b})$, $a_{p_1} = 2b_{p_1} + 1 < \infty$;
 - (ii) $\Pi_1(\bar{a}, \bar{b}) \cup \Pi_q(\bar{a}, \bar{b})$ equals the set of those $p \in \mathbb{P} \setminus \Pi'_1(\bar{a}, \bar{b})$, for which the exponent of q modulo p is not divisible by any $\pi \in \mathbb{P} \setminus (\Pi_1(\bar{a}, \bar{b}) \cup \Pi'_1(\bar{a}, \bar{b}))$;
 - (iii) $b_{p'} \neq 0$, for any $p' \in \mathbb{P} \setminus (\Pi_1(\bar{a}, \bar{b}) \cup \Pi'_1(\bar{a}, \bar{b}) \cup \Pi(\bar{a}, \bar{b}))$, and $a_{p_2} \leq 2b_{p_2} < \infty$, for each $p_2 \in \Pi_q(\bar{a}, \bar{b}) \setminus \Pi(\bar{a}, \bar{b})$; in addition, if $2 \leq a_{\bar{p}_2} = 2b_{\bar{p}_2}$, for some $\bar{p}_2 \in \Pi_q(\bar{a}, \bar{b})$, then $\mathbb{P} \setminus \Pi_1(\bar{a}, \bar{b})$ contains at least two elements;
 - (iv) $q \notin \Pi_1(\bar{a}, \bar{b}) \cup \Pi_q(\bar{a}, \bar{b})$, and if $q \notin \Pi(\bar{a}, \bar{b})$, then $a_q = b_q + 1$.
- Then there exists a field E with $\text{char}(E) = q$ and $(\text{abrd}_p(E), \text{Brd}_p(E)) = (a_p, b_p)$, for each $p \in \mathbb{P}$.

To prove Proposition 5.6 and the latter assertion of Theorem 1.1 in a unified way, we put $\Pi_1(\bar{a}, \bar{b}) = \Pi'_1(\bar{a}, \bar{b}) = \infty$, fix a number $q \in \mathbb{P}$, and under the hypotheses of Theorem 1.1, denote by $\Pi_q(\bar{a}, \bar{b})$ the set of prime divisors of $q - 1$. We show that the desired field E can be chosen so as to possess a Henselian valuation v with \widehat{E} and $v(E)$ subject to the following restrictions:

- (6.3) (i) $\Pi_1(\bar{a}, \bar{b}) \cup \Pi'_1(\bar{a}, \bar{b})$ consists of those $p \in \mathbb{P}$, for which \widehat{E} includes as a subfield a \mathbb{Z}_p -extension of its prime subfield; $\Pi_q(\bar{a}, \bar{b}) \cup \Pi_1(\bar{a}, \bar{b})$ equals the set of those $p \in \mathbb{P}$, for which \widehat{E} contains a primitive p -th root of unity;
- (ii) $\tau(p) = \infty$ if and only if $a_p = \infty$; $\tau(p) = a_p$, provided that $a_p = 0$ or p lies in the union of $\Pi_1(\bar{a}, \bar{b})$ and $\Pi_q(\bar{a}, \bar{b}) \setminus \Pi(\bar{a}, \bar{b})$; when $a_p = 0$ or $p \in \Pi_q(\bar{a}, \bar{b}) \setminus \Pi(\bar{a}, \bar{b})$ and $a_p = 2b_p$, $\mathcal{G}(\widehat{E}(p)/\widehat{E})$ is isomorphic to \mathbb{Z}_p ;
- (iii) $2b_\pi - 1 \leq \tau(\pi) \leq 2b_\pi$ and \mathbb{Z}_π is isomorphic to $\mathcal{G}(\widehat{E}(\pi)/\widehat{E})$ and the Sylow pro- π -subgroups of $\mathcal{G}_{\widehat{E}}$, provided that $\pi \in \Pi(\bar{a}, \bar{b}) \cap \Pi_q(\bar{a}, \bar{b})$ and $a_\pi < \infty$; $r_{\pi_q}(\widehat{E}) = 2(b_{\pi_q} - \lfloor a_{\pi_q}/2 \rfloor)$ in case $\pi_0 \in \Pi_q(\bar{a}, \bar{b}) \setminus \Pi(\bar{a}, \bar{b})$ and $a_{\pi_q} < 2b_{\pi_q}$;
- (iv) $r_q(\widehat{E}) = b_q = \tau(q)$ in case $q \in \Pi(\bar{a}, \bar{b})$ and $b_q > 0$;
- (v) If $p \notin \Pi(\bar{a}, \bar{b}) \cup \Pi_1(\bar{a}, \bar{b}) \cup \Pi_q(\bar{a}, \bar{b})$, then $r_p(\widehat{E}) = b_p$ and $\tau(p) = a_p$.

Moreover, we fix (E, v) so as to satisfy (6.3) and the following conditions:

- (6.4) (i) If $b_{\pi'} = 0 \neq a_{\pi'}$, for each $\pi' \notin \Pi_1(\bar{a}, \bar{b}) \cup \{p\}$, where $p = q$, or p is an element of $\Pi_q(\bar{a}, \bar{b}) \setminus \Pi(\bar{a}, \bar{b})$ satisfying the equality $a_p = 2b_p$, then $\mathcal{G}_{\widehat{E}}$ is a Frattini cover of the group $\mathcal{G}(\widehat{E}(p)/\widehat{E}) \times \text{Alt}_\infty$;
- (ii) If $b_{\bar{\pi}} \neq 0$ or $a_{\bar{\pi}} = 0$, for some $\bar{\pi} \notin \Pi_1(\bar{a}, \bar{b}) \cup \{q\}$, then each $\pi \in \Pi_1(\bar{a}, \bar{b})$ satisfies the equality $\tau(\pi) = a_\pi$ and the condition stated in (6.2) (iii); in addition, if $(a_q, b_q) = (1, 0)$, then there is a finite Galois extension \widehat{E}'_q of \widehat{E}

in \widehat{E}_{sep} , such that $\mathcal{G}(\widehat{E}'_q/\widehat{E})$ is an NMM-group whose Sylow q -subgroup is normal of exponent q and prime index;

(iii) If $p \notin \Pi(\bar{a}, \bar{b})$ and $b_{\bar{p}} \neq 0$ or $a_{\bar{p}} = 0$, for some $\bar{p} \notin \Pi_1(\bar{a}, \bar{b}) \cup \{p\}$, then \widehat{E} has a finite Galois extension \widehat{E}'_p in \widehat{E}_{sep} , such that $\mathcal{G}(\widehat{E}'_p/\widehat{E})$ is an NMM-group with a normal Sylow p -subgroup of exponent p and prime index.

The existence of fields subject to the restrictions of (6.1) and (6.2), or (6.3) and (6.4), is implied by Lemma 2.2 and the following two lemmas.

Lemma 6.3. *Assume that E_0 is a field such that \mathcal{G}_{E_0} is procyclic, and let H be a pronilpotent group with $\text{cd}(H) \leq 1$ and $r_p(H) \geq r_p(E_0)$, for every $p \in \mathbb{P}$. Then there exists a field extension E/E_0 , such that $\mathcal{G}_E \cong H$ and E_0 is algebraically closed in E .*

Proof. Let H_p be the Sylow pro- p -subgroup of H , for each $p \in \mathbb{P}$, $P_{\infty}(H) = \{p \in \mathbb{P}: r_p(H_p) = \infty\}$, and $P_{\text{ncyc}}(H) = \{p \in \mathbb{P} \setminus P_{\infty}(H): r_p(H_p) \geq 2\}$. Denote by Θ_p some elementary abelian p -group of rank $r_p(H_p) - 1$, for each $p \in P_{\text{ncyc}}(H)$, fix group products $\Theta = \prod_{p \in P_{\text{ncyc}}(H)} \Theta_p$, $H_0 = \prod_{p \in P_{\infty}(H)} H_p$, $H_1 = \Theta \times H_0$ and $\overline{H} = \mathcal{G}_{E_0} \times H_1$, and let H be a Frattini cover of \overline{H} . It is known (cf. [51]) that there exist extensions E_1 and E'_1 of E_0 , such that E'_1/E_0 is purely transcendental, $E_1 \in I(E'_1/E_0)$ and E'_1/E_1 is Galois with $\mathcal{G}(E'_1/E_0) \cong H_1$. Identifying $E_{0,\text{sep}}$ with its E_0 -isomorphic copy in $E'_{1,\text{sep}}$, and observing that E_0 is algebraically closed in E'_1 , one obtains that $E_{0,\text{sep}}E'_1/E_1$ is Galois with $\mathcal{G}(E_{0,\text{sep}}E'_1/E_1) \cong \overline{H}$. Hence, by Lemma 2.3, there is an extension E/E_1 , such that $L = E_{0,\text{sep}}E'_1 \otimes_{E_1} E$ is a field, $\text{cd}(\mathcal{G}_E) \leq 1$, and $L \cap R \neq E$, for every $R \in I(L_{\text{sep}}/E)$, $R \neq E$. This ensures that L/E is a Galois extension with $\mathcal{G}(L/E) \cong \overline{H}$, and \mathcal{G}_E is a Frattini cover of \overline{H} . The obtained results also indicate that E_0 and E_1 are algebraically closed in E , and there is an E -isomorphism $L_{\text{sep}} \cong E_{\text{sep}}$. Since H is pronilpotent with $\text{cd}(H) \leq 1$, they imply in conjunction with Lemma 2.4 that $\mathcal{G}_E \cong H$, which completes the proof of Lemma 6.3. \square

Lemma 6.4. *Let E be a field with $\text{char}(E) = q \geq 0$, \mathcal{G}_E pronilpotent and $\text{cd}(\mathcal{G}_E) = 1$. Suppose that Π is a nonempty proper subset of \mathbb{P} , E contains a primitive π -th root of unity, for each $\pi \in \Pi \setminus \{q\}$, and $\bar{p} \notin \Pi$ in case $P(\mathcal{G}_E) = \{\bar{p}\}$. Then there exists a field extension E'/E satisfying the following:*

(i) *E is algebraically closed in E' , $\text{cd}(\mathcal{G}_{E'}) = 1$, and $r_p(E') = r_p(E)$, for every $p \in \mathbb{P}$;*

(ii) *There exist finite Galois extensions M_{π} , $\pi \in \Pi$, of E' in E'_{sep} , such that $\mathcal{G}(M_{\pi}/E')$ is an NMM-group with a normal Sylow π -subgroup G_{π} of exponent π , for each π ; in addition, if $P(\mathcal{G}_E)$ contains the prime divisors of $\pi - 1$, then G_{π} is noncyclic;*

(iii) *$\mathcal{G}_{E'}$ is a Frattini cover of $\mathcal{G}(\Psi/E)$, where Ψ is the compositum of the fields E' , $E(p)$, $p \in P(\mathcal{G}_E)$, and M_{π} , $\pi \in \Pi$.*

Proof. Let $E(X)$ be a rational function field in one indeterminate over E . Then it follows from (2.9), its analogue in characteristic q , and our assumptions that there exists a set L_{π} , $\pi \in \Pi$, of finite Galois extensions of $E(X)$

in $E(X)_{\text{sep}}$, such that $\mathcal{G}(L_\pi/E(X))$ is an NMM-group with a normal Sylow π -subgroup H_π of exponent π and index $\pi' \in \mathbb{P}$, and $I(L_\pi/E(X))$ contains as an element a degree π' cyclic extension of $E(X)$, for each index π . In view of Galois theory, this ensures the existence of a field extension E'/E with the properties required by Lemma 6.4. \square

Let now (E, v) be a Henselian equicharacteristic field subject to the restrictions of (6.1) and (6.2), or (6.3) and (6.4), depending on whether $\text{char}(\widehat{E}) = 0$ or $\text{char}(\widehat{E}) = q > 0$. Then it follows from Theorem 1.3, Proposition 6.1 and the assumptions of Theorems 1.1, 1.2 and Proposition 5.6 that $(\text{abrd}_p(E), \text{Brd}_p(E)) = (a_p, b_p)$, for each $p \in \mathbb{P}$ different from q . This proves our assertion when $q = 0$, so we assume further that $q > 0$. In this case, one may consider only the special case in which $b_q \leq a_q \leq b_q + 1 < \infty$ (since Remark 3.3 indicates that $\text{Brd}_q(E) = \infty$ whenever $\tau(q) = \infty$). Applying Proposition 5.1 and analyzing the proof of Lemma 2.1 in [7], one concludes that (E, v) can be defined as an algebraic extension of an iterated Laurent formal power series field $E_\nu = Y((X_1)) \dots ((X_\nu))$ in $\nu = \tau(q) = b_q + 1$ indeterminates over a perfect field Y , which satisfies the following conditions:

- (6.5) (i) $(\text{abrd}_p(Y), \text{Brd}_p(Y)) = (a_p, b_p)$, for all $p \in \mathbb{P} \setminus \{q\}$;
- (ii) Finite extensions of E_ν in E are tamely and totally ramified relative to the standard \mathbb{Z}' -valued valuation w_ν of E , acting trivially on Y ;
- (iii) $w(E)/qw(E)$ is of order $q^{\tau(q)}$ and $w(E) = pw(E)$, for every $p \in \mathbb{P} \setminus \{q\}$, where w is a valuation of E extending w_ν .

By Krull's theorem, (E_ν, w_ν) is maximally complete; in particular, (E_ν, w_ν) is Henselian and it follows from (6.5) (i) that E_ν is isomorphic to its finite extensions in E . In view of (1.1) and [6], (1.2), these observations indicate that $\text{Brd}_q(E) = \text{Brd}_q(E_\nu)$ and $\text{abrd}_q(E) = \text{abrd}_q(E_\nu)$. Also, it follows from (6.5) (iii) and Proposition 5.1 that $\nu - 1 \leq \text{Brd}_q(E_\nu) \leq \text{abrd}_q(E_\nu) \leq \nu$. Summing-up (6.3), (6.4), (6.5) and Proposition 5.1, one concludes that $(\text{abrd}_q(E), \text{Brd}_q(E)) = (\text{abrd}_q(E_\nu), \text{Brd}_q(E_\nu)) = (a_q, b_q)$. Finally, it follows from (3.1) (ii) and (6.5) (i), (iii), that $(\text{abrd}_p(E), \text{Brd}_p(E)) = (\text{abrd}_p(Y), \text{Brd}_p(Y)) = (a_p, b_p)$ whenever $p \in \mathbb{P} \setminus \{q\}$, which completes the proofs of Theorem 1.1 and Proposition 6.2.

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